

TOTALLY GEODESIC SPECTRA OF ARITHMETIC HYPERBOLIC SPACES

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ABSTRACT. In this paper we show that totally geodesic subspaces determine the commensurability class of a standard arithmetic hyperbolic n -orbifold, $n \geq 4$. Many of the results are more general and apply to locally symmetric spaces associated to arithmetic lattices in \mathbb{R} -simple Lie groups of type B_n and D_n . We use a combination of techniques from algebraic groups and quadratic forms to prove several results about these spaces.

1. INTRODUCTION

The goal of this paper is to determine the extent to which the geometry of an arithmetic hyperbolic n -manifold, $n \geq 4$, is encoded in the collection of its totally geodesic submanifolds. To put this goal in a broader context, we step back a moment and ask a natural question, one going back over a century: *What topological and geometric properties of a space M are encoded in certain interesting collections of geometric data associated to M ?* One of the earliest examples of this line of inquiry was in 1911, when Weyl showed that the eigenvalues of the Laplace-Beltrami operator determine the dimension and volume of a closed Riemannian manifold [42]. In 1966, Kac popularized this question by asking “Can one hear the shape of a drum?” [16]. Since that time many different collections of data, often called **spectra**, have been studied. Over the past few decades, one prominent spectrum has been the collection of lengths of closed geodesics. The **weak length spectrum** (sometimes also referred to as the **length set**) of a Riemannian manifold M , is the set

$$(1.1) \quad L(M) := \{\lambda \in \mathbb{R} \mid \lambda \text{ is the length of a closed geodesic in } M\}.$$

Observe that this collection can be equivalently formulated as

$$(1.2) \quad L(M) = \{\text{Isometry classes of closed geodesics in } M\}.$$

We call two manifolds with the same weak length spectrum **weakly iso-length-spectral**.

Question 1. *Are weakly iso-length-spectral spaces necessarily isometric?*

The answer is a resounding no, and since the 1960’s, there have been many constructions of weakly iso-length-spectral spaces that are not isometric, the most famous of which being:

- 16-dimensional flat tori (Milnor, 1964 [27]),
- 2- and 3-dimensional hyperbolic manifolds, and more generally spaces coming from quaternion algebras (Vignéras, 1980 [39]),
- General method based on covering space theory (Sunada, 1985 [35]).

However, these constructions produce manifolds that are *almost* isometric in the sense that they are commensurable (Section 2). When two Riemannian manifolds M_1 and M_2 are commensurable, the length of every closed geodesic in M_1 is a rational multiple of the length of a closed geodesic in M_2 , and vice versa. Motivated by this, [9] defined the **rational length spectrum** to be the set

$$(1.3) \quad \mathbb{Q}L(M) := \{s\lambda \in \mathbb{R} \mid s \in \mathbb{Q} \text{ and } \lambda \text{ is the length of a closed geodesic in } M\}.$$

Again, we observe that this definition may be reformulated as follows:

$$(1.4) \quad \mathbb{Q}L(M) = \{\text{Commensurability classes of closed geodesic in } M\}.$$

Two manifolds with the same rational length spectrum are said to be **length-commensurable**. In particular, commensurable manifolds are length-commensurable. One may then ask the following refined question.

Question 2. *Are length-commensurable spaces necessarily commensurable?*

In many cases, the answer is yes. When M_1 and M_2 are arithmetic hyperbolic n -manifolds, then $\mathbb{Q}L(M_1) = \mathbb{Q}L(M_2)$ implies M_1 and M_2 are commensurable in each of the following cases:

- $n = 2$ (Reid, 1992 [32]),
- $n = 3$ (Chinburg, Hamilton, Long, and Reid, 2008 [9]),
- $n \neq 3, n \neq 7, n \not\equiv 1 \pmod{4}$ (Prasad and Rapinchuk, 2009 [30]),
- $n = 7$ (Garibaldi, 2013 [12]).

However, for each positive $n \equiv 1 \pmod{4}$, $n > 1$, [30] produced examples of noncommensurable length-commensurable arithmetic hyperbolic n -manifolds. More generally, there are many constructions of families of pairwise noncommensurable length-commensurable arithmetic locally symmetric spaces of the same Killing–Cartan type (see [20, Theorem 1], [30, Construction 9.15]).

Our motivation is to find a collection of data that is complementary to length spectra and that distinguishes commensurability classes. In recent years, there have been many papers looking at certain higher dimensional analogues of geodesics: totally geodesic subspaces. For us, it will be sufficient to only consider *nonflat* totally geodesic subspaces. Furthermore, in analogy with looking at closed geodesics, we only want to look at *finite volume* subspaces. With this in mind, we define the **totally geodesic set** of a Riemannian manifold to be the set

$$(1.5) \quad TG(M) := \left\{ \begin{array}{c} \text{Isometry classes of nonflat finite volume} \\ \text{totally geodesic subspaces of } M \end{array} \right\}.$$

McReynolds and Reid [25] prove that if M_1 and M_2 are standard [22, Theorem 10.2.3] arithmetic hyperbolic 3-manifolds such that $TG(M_1) = TG(M_2)$, then M_1 and M_2 are commensurable. As was the case for the weak length spectrum, $TG(M)$ is not an invariant of commensurability class, and hence we define the **rational totally geodesic spectrum** to be the set

$$(1.6) \quad \mathbb{Q}TG(M) := \left\{ \begin{array}{c} \text{Commensurability classes of nonflat finite volume} \\ \text{totally geodesic subspaces of } M \end{array} \right\}.$$

Observe that $TG(M)$ and $\mathbb{Q}TG(M)$ are natural analogues of the second formulations of $L(M)$ and $\mathbb{Q}L(M)$ (see 1.2 and 1.4). The former is more rigid while the later is an invariant of the commensurability class of M . If two Riemannian orbifolds M and M' have the same rational totally geodesic spectrum, we say they are **totally-geodesic-commensurable**. The goal of this paper is to investigate the following question:

Question 3. *Are totally-geodesic-commensurable spaces necessarily commensurable?*

In this paper we address this question in the case of locally symmetric spaces of type B_n and D_n . In particular, we focus on standard arithmetic locally symmetric spaces associated to Lie groups of the form $\prod_{i=1}^r \mathbf{SO}(p_i, m - p_i) \times (\mathbf{SO}_m(\mathbb{C}))^s$, for $m \geq 5$. These spaces are constructed via the isometry groups of quadratic forms over number fields (see Construction 4.8). We call a locally symmetric space **\mathbb{R} -simple** if its associated Lie group is not a nontrivial product (i.e., if $r + s = 1$). Note that standard arithmetic hyperbolic n -manifolds are \mathbb{R} -simple.

The first step to proving that the rational totally geodesic spectrum determines the commensurability class is showing it determines the field of definition, which we do in Section 6

Theorem A. *Let M_1 and M_2 be \mathbb{R} -simple, arithmetic, locally symmetric orbifolds coming from quadratic forms of dimension $m \geq 5$ over number fields k_1 and k_2 respectively. If $\mathbb{Q}TG(M_1) = \mathbb{Q}TG(M_2)$, then k_1 and k_2 are isomorphic.*

Using the technical results of Section 7 on quadratic forms and their isometry groups, we are then able to prove our main theorem on commensurability.

Theorem B. *Let M_1 and M_2 be \mathbb{R} -simple, arithmetic locally symmetric orbifolds coming from quadratic forms of dimension $m \geq 5$. If $\text{QTG}(M_1) = \text{QTG}(M_2)$, then M_1 and M_2 are commensurable.*

Unlike [30] and [12], Theorem B is not dependent in $\mathbb{R}\text{-rank} \geq 2$ upon the truth of Schanuel's conjecture. Specializing to the \mathbb{R} -rank one case, we obtain a result about standard arithmetic hyperbolic orbifolds.

Theorem C. *Let M_1 and M_2 be standard arithmetic hyperbolic n -orbifolds, $n \geq 4$. If $\text{QTG}(M_1) = \text{QTG}(M_2)$, then M_1 and M_2 are commensurable.*

In fact, we show that codimension one and codimension two totally geodesic subspaces determine the commensurability class of a standard arithmetic hyperbolic orbifold. We go on to show in Theorem 8.2 that the commensurability class of an even dimensional arithmetic hyperbolic orbifold is totally determined by its codimension one totally geodesic subspaces. To complement these results, we then show that there are many commensurability classes of hyperbolic orbifolds with the exact same collection of totally geodesic subspaces in codimension greater than 2.

Theorem D (Hyperbolic Subspace Dichotomy). *Let M_1 and M_2 be standard arithmetic hyperbolic n -orbifolds, $n \geq 4$, with fields of definition k_1 and k_2 , respectively. Then either*

- (i) $k_1 \cong k_2$, in which case, for all $j \in \mathbb{N}$, $1 < j < n - 2$, up to commensurability, M_1 and M_2 have the exact same collection of j -dimensional finite volume totally geodesic subspaces, or
- (ii) $k_1 \not\cong k_2$, in which case, up to commensurability, M_1 and M_2 do not share a single finite volume totally geodesic subspace of dimension ≥ 2 .

The techniques that we use to prove Theorems A - D have many applications. In Section 9 we use them to answer a question posed to us by Jean-François Lafont, and in Appendix A we give an alternate proof of Machlachlan's parametrization of commensurability classes of even dimensional arithmetic hyperbolic orbifolds.

Along the way, we construct several explicit examples of commensurability classes of standard arithmetic hyperbolic orbifolds with specific properties. In particular, in Example 9.5, we construct a hyperbolic 3-orbifold N and a hyperbolic 5-orbifold M such that every totally geodesic surface in N is commensurable to a totally geodesic surface in M , yet N is not commensurable to a totally geodesic subspace of M .

While all even dimensional arithmetic hyperbolic orbifolds come from quadratic forms, there are odd dimensional ones that do not. To this end we also address some results on spaces coming from skew Hermitian forms over quaternion division algebras over number fields. Though there are considerable obstructions to finishing the analysis for groups coming from this construction, we do have initial results. To start, an \mathbb{R} -simple, nonstandard arithmetic, locally symmetric space cannot be totally-geodesic-commensurable with a standard one.

Theorem E. *Let M_1 and M_2 be \mathbb{R} -simple, arithmetic locally symmetric spaces where M_1 comes from a quadratic form and M_2 comes from a skew Hermitian form over a division algebra. Then $\text{QTG}(M_1) \neq \text{QTG}(M_2)$.*

Furthermore, the rational totally geodesic spectrum determines the field and algebra of definition of a nonstandard arithmetic lattice.

Theorem F. *Let M_1 and M_2 be \mathbb{R} -simple arithmetic locally symmetric spaces coming from skew Hermitian forms of dimension $n \geq 4$ over quaternion division algebras D_1 and D_2 over number fields k_1 and k_2 respectively. If $\text{QTG}(M_1) = \text{QTG}(M_2)$, then k_1 and k_2 are isomorphic and this isomorphism induces an isomorphism between D_1 and D_2 .*

2. NOTATION AND PRELIMINARY RESULTS:
COMMENSURABILITY, TOTALLY GEODESIC SUBSPACES, AND LOCALLY SYMMETRIC SPACES

In this paper F is a field that is not of characteristic 2, \overline{F} is a fixed algebraic closure of F , k is a number field, and \mathcal{O}_k is its ring of integers.

Two subgroups Γ_1 and Γ_2 of a group G are **commensurable** if $\Gamma_1 \cap \Gamma_2$ is finite index in both Γ_1 and Γ_2 . Following [30], we shall say two subgroups Γ_1, Γ_2 of a G are **commensurable up to G -automorphism** if there exists a G -automorphism φ such that Γ_1 and $\varphi(\Gamma_2)$ are commensurable. (Note that some authors refer to this notion as **commensurable in the wide sense** [22, Def. 1.3.4].) Commensurability up to G -automorphism is an equivalence relation among subgroups of G . Two Riemannian manifolds are **commensurable** if they have isometric finite sheeted covers.

Let M be a Riemannian manifold and let $N \subset M$ be a connected immersed submanifold. Recall that N is **geodesic at $p \in N$** if every geodesic of M starting at p and tangent to N at p is a geodesic of N . If N is geodesic at each of its points it is called **totally geodesic**.

Following [37, Chp 13], we call the quotient of a manifold by a properly discontinuous (not necessarily free) group action a **good orbifold**. Since discrete subgroups of semisimple Lie groups often have torsion, good orbifolds naturally appear in the commensurability classes of locally symmetric manifolds. When a good orbifold is a quotient of a Riemannian manifold by isometries, we call it a good Riemannian orbifold. Every good Riemannian orbifold naturally has a Riemannian manifold universal cover. A subspace of a Riemannian orbifold is defined to be totally geodesic if it is the image of a totally geodesic subspace in its universal cover. It follows that the sets $TG(M)$ and $\mathbb{Q}TG(M)$ (Definitions 1.5 and 1.6) make sense for all good Riemannian orbifolds.

Lemma 2.1. *Commensurable good Riemannian orbifolds are totally-geodesic-commensurable.*

Proof. Let M_1 and M_2 be commensurable and \widetilde{M} be a shared finite sheeted cover with projections π_1 and π_2 . Pick a nonflat finite volume totally geodesic subspace $N_1 \subset M_1$. Then $N_2 := \pi_2(N')$, where N' is a connected component of $\pi_1^{-1}(N_1)$, is a totally geodesic submanifold of M_2 . Since π_1 and π_2 are finite sheeted covers, N_2 is also nonflat and of finite volume. By symmetry of argument, the result follows. \square

In general, totally geodesic subspaces are rare, and we should only expect to find such subspaces when we are considering an ambient space with many symmetries. As such, in what follows, we shall only consider locally symmetric spaces. A Riemannian manifold M is a **globally symmetric space** if each point $p \in M$ is an isolated fixed point of an involutive isometry of M . Totally geodesic subspaces of a globally symmetric space are also globally symmetric [15, Ch. IV Prop 7.1]. One of the advantages to working with globally symmetric spaces is that questions about the spaces can be translated into questions about its isometry group. A globally symmetric space is of **noncompact type** if $G := \text{Isom}^\circ(M)$ is a semisimple Lie group with no compact factors, in which case M is isometric to G/K where K is a maximal compact subgroup of G .

Lemma 2.2. *Let M a connected globally symmetric space of noncompact type, $G = \text{Isom}^\circ(M)$ and K a stabilizer of a point $p_0 \in M$.*

- (i) *Let $H \subset G$ be a semisimple Lie subgroup with no compact factors. Then $N_H := H/(H \cap K)$ is a totally geodesic submanifold of M .*
- (ii) *Let $N \subset M$ be a totally geodesic submanifold of noncompact type such that $p_0 \in N$. Then there exists a semisimple Lie subgroup $H_N \subset G$ with no compact factors such that $H_N/(H_N \cap K) = N$.*

Proof.

(1). Note that N_H is an immersed submanifold of M . Geodesics of M arise from the exponential map of G . Given an element $X \in \text{Lie}(H)$ we know that $\exp_G(tX) \in H$ for all $t \in \mathbb{R}$, and hence N must be totally geodesic.

(2). Let $\text{Lie}(G) = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition. Let $\mathfrak{s} \subset \mathfrak{p}$ be the subspace associated with the tangent space of N . Then \mathfrak{k} acts on \mathfrak{p} by the adjoint representation and let $\mathfrak{k}' = N_{\mathfrak{k}}(\mathfrak{s}) = \{X \in \mathfrak{k} \mid \text{ad}(X)(\mathfrak{s}) \subset \mathfrak{s}\}$. Then $\mathfrak{h} := \mathfrak{k}' \oplus \mathfrak{s}$ is a Lie subalgebra of $\text{Lie}(G)$. Let H_N be the unique connected Lie subgroup of G with Lie algebra \mathfrak{h} . It follows that H_N has the desired properties. \square

A good Riemannian orbifold M is a **locally symmetric space** if M has universal cover \widetilde{M} that is a globally symmetric space. In which case $M = \Gamma \backslash \widetilde{M}$ where Γ is a discrete subgroup of $\text{Isom}^\circ(\widetilde{M})$. A locally symmetric space is of **noncompact type** if its universal cover is a globally symmetric space of noncompact type. Totally geodesic subspaces of a locally symmetric space are also locally symmetric. The study of locally symmetric spaces of noncompact type translates to the study of discrete subgroups of semisimple Lie groups with no compact factors, as we shall now record with the following well known proposition.

Proposition 2.3. *Let $M_1 = \Gamma_1 \backslash G_1 / K_1$ and $M_2 = \Gamma_2 \backslash G_2 / K_2$ be locally symmetric spaces of noncompact type where G_1 and G_2 are connected, adjoint, semisimple Lie groups with no compact factors. Then M_1 and M_2 are isometric if and only if there is a Lie group isomorphism $\varphi : G_1 \rightarrow G_2$ such that $\varphi(K_1) = K_2$ and $\varphi(\Gamma_1) = \Gamma_2$*

Since the image of a maximal compact (resp. discrete) subgroup under an automorphism is always a maximal compact (resp. discrete) subgroup, understanding isometry classes of locally symmetric spaces of noncompact type with universal cover G/K reduces to understanding $\text{Aut}(G)$ -orbits of discrete subgroups of G . In particular, understanding the commensurability classes of locally symmetric spaces is equivalent to understanding the commensurability classes of discrete subgroups of G up to G -automorphism.

Let G be a semisimple Lie group and $\Gamma \subset G$ be a discrete subgroup. The Haar measure on G naturally descends to a G -invariant measure on $\Gamma \backslash G$. When the Haar measure on G descends to a measure of finite volume on $\Gamma \backslash G$, Γ is called a **lattice**. When $\Gamma \backslash G$ is compact, Γ is said to be **cocompact** or a **uniform lattice**. Cocompact discrete subgroups are always lattices. A lattice is **irreducible** if, up to commensurability, it is not a product of smaller lattices. Being cocompact, a lattice, or irreducible is an invariant of commensurability class.

Henceforth, our orbifolds will be good and our locally symmetric spaces will be of noncompact type.

3. ARITHMETIC GROUPS AND ARITHMETIC LOCALLY SYMMETRIC SPACES

Arithmetic Subgroups of Algebraic \mathbb{Q} -Groups.

Let \mathbf{G} be an algebraic group defined over \mathbb{Q} . There exists a faithful \mathbb{Q} -rational embedding $\rho : \mathbf{G} \rightarrow \mathbf{GL}(V)$ for some \mathbb{Q} -vector space V [2, 1.10]. Let $L \subset V$ be a \mathbb{Z} -lattice of V , i.e., a free \mathbb{Z} -module such that $L \otimes_{\mathbb{Z}} \mathbb{Q} = V$. Define the group

$$G_{\rho,L} := \{g \in \mathbf{G}(\mathbb{Q}) \mid \rho(g)(L) = L\}.$$

Any subgroup $\Gamma \subset \mathbf{G}(\mathbb{Q})$ commensurable with $G_{\rho,L}$ is an **arithmetic subgroup** of $\mathbf{G}(\mathbb{Q})$. Were we to chose a different embedding, ρ' , and different \mathbb{Z} -lattice, L' , we would have obtained a different group $G_{\rho',L'}$, however, any such $G_{\rho',L'}$ is commensurable with $G_{\rho,L}$ (see [3, 7.12] and preceding discussion). It follows that the commensurability class of an arithmetic group is independent of the choices of ρ and L . In other words, the \mathbb{Q} -isomorphism class of \mathbf{G} determines a commensurability class of arithmetic groups.

Often we will assume the existence of some embedding ρ and lattice L , and we will denote $\mathbf{G}(\mathbb{Z}) := G_{\rho,L}$. Note however that not all arithmetic groups arise as the stabilizer of a lattice. This can be seen from the fact that every lattice stabilizer contains a congruence subgroup [3, 7.12] but there are arithmetic groups that do not contain any congruence subgroups (for example, there are such groups in $\mathbf{SL}_2(\mathbb{Z})$) [31, §2.1].

One way to construct algebraic \mathbb{Q} -groups is to start with a k -group, where k is a number field, and then apply the **Weil restriction of scalars functor** $R_{k/\mathbb{Q}}$ [29, §2.1.2], [22, §10.3]. This functor has the property that if \mathbf{G} is an algebraic k -group, then $R_{k/\mathbb{Q}}\mathbf{G}$ is an algebraic \mathbb{Q} -group and there is an abstract group isomorphism between $\mathbf{G}(k)$ and $(R_{k/\mathbb{Q}}\mathbf{G})(\mathbb{Q})$. With this identification, it makes sense to talk about arithmetic subgroups of $\mathbf{G}(k)$. Furthermore, it is not hard to see that arithmetic subgroups of $\mathbf{G}(k)$ are precisely the groups commensurable with the stabilizer of an \mathcal{O}_k -lattice of a k -vector space V where there is a k -rational embedding of \mathbf{G} into $\mathbf{GL}(V)$.

An **absolutely (resp. absolutely almost) simple algebraic F -group** is an algebraic F -group that, upon extending scalars to \overline{F} , is (resp. isogenous to) a simple semisimple algebraic \overline{F} -group. For example the \mathbb{C} -group \mathbf{SL}_n is absolutely almost simple but not absolutely simple since it has nontrivial center equal to the group of n^{th} roots of unity. The semisimple \mathbb{R} -group $R_{\mathbb{C}/\mathbb{R}}\mathbf{SL}_2$, which is related to the study of hyperbolic 3-manifolds, is not absolutely almost simple, since it is \mathbb{C} -isomorphic to $\mathbf{SL}_2 \times \mathbf{SL}_2$. If we start with an absolutely almost simple k -group, then $R_{k/\mathbb{Q}}(\mathbf{G})$ is always a semisimple \mathbb{Q} -group. An **F -simple F -group** is an algebraic F -group which, up to isogeny, does not contain a proper nontrivial normal F -subgroup. Absolutely almost simple F -groups are F -simple and $R_{\mathbb{C}/\mathbb{R}}\mathbf{SL}_2$ is \mathbb{R} -simple. All semisimple k -groups are built from absolutely almost simple groups over number fields [7, 6.21(ii)] and [10, Prop. A.5.14]. For the reader's convenience, we record a corollary of [10, Prop. A.5.14] that will be useful in what follows.

Proposition 3.1. *Let \mathbf{G} be a semisimple k -simple k -group.*

- (i) *(Existence) There exists a number field k' containing k and an absolutely almost simple k' -group \mathbf{H}' such that \mathbf{G} and $R_{k'/k}(\mathbf{H}')$ are k -isogenous. Furthermore, if \mathbf{G} is adjoint, \mathbf{G} and $R_{k'/k}(\mathbf{H}')$ are k -isomorphic.*
- (ii) *(Uniqueness) The pair (\mathbf{H}', k') is unique in the following sense: If k'' is a number field containing k , and \mathbf{H}'' an absolutely almost simple k'' -group such that \mathbf{G} and $R_{k''/k}\mathbf{H}''$ are k -isogenous, then there is a field isomorphism $\tau : k' \rightarrow k''$ and a k'' -isogeny between $\mathbf{H}' \times_{\tau} \text{spec } k'$ and \mathbf{H}'' .*

Arithmetic Lattices in Semisimple Lie Groups.

Let \overline{G} be a connected, adjoint, semisimple Lie group with no compact factors. Let $\Gamma \subset \overline{G}$ be a lattice. Then Γ is **arithmetic** if there exists a semisimple algebraic \mathbb{Q} -group \mathbf{G} and a surjective analytic homomorphism $\pi : \mathbf{G}(\mathbb{R})^\circ \rightarrow \overline{G}$ with compact kernel such that $\pi(\mathbf{G}(\mathbb{Z}) \cap \mathbf{G}(\mathbb{R})^\circ)$ and Γ are commensurable up to \overline{G} -automorphism.

$$\begin{array}{ccccc} \mathbf{G}(\mathbb{Z}) \cap \mathbf{G}(\mathbb{R})^\circ & \xrightarrow{\quad\quad\quad} & \mathbf{G}(\mathbb{R})^\circ & \longrightarrow & \mathbf{G}(\mathbb{R}) \\ \pi \downarrow & & \downarrow \pi & & \\ \pi(\mathbf{G}(\mathbb{Z}) \cap \mathbf{G}(\mathbb{R})^\circ) & \xleftarrow{\sim_c} \varphi(\Gamma) & \longrightarrow & \overline{G} & \end{array}$$

In what follows, we shall say that \mathbf{G} **gives rise to** Γ . If $\mathbf{H} \subset \mathbf{G}$ is a \mathbb{Q} -simple factor, we will always assume that it is \mathbb{R} -isotropic, since otherwise $\mathbf{H}(\mathbb{R})^\circ \subset \ker(\pi)$, and we may just replace \mathbf{G} with \mathbf{G}/\mathbf{H} . Observe that if $\Gamma, \Gamma' \subset \overline{G}$ are subgroups that are commensurable up to \overline{G} -automorphism and one is an arithmetic lattice, then so is the other.

It may appear as though arithmetic lattices are rather specific and potentially rare type of lattice. However, thanks to Margulis's arithmeticity theorem [23] and the work of Gromov and Schoen [14], irreducible lattices in groups not locally isomorphic to $\mathbf{SO}(n, 1)$ or $\mathbf{SU}(n, 1)$ are always arithmetic.

Arithmetic Locally Symmetric Spaces.

In this section we adopt the following notation:

- \overline{G} is a connected, adjoint, semisimple Lie group with no compact factors,
- $\overline{K} \subset \overline{G}$ is a maximal compact subgroup,
- \mathbf{G} is a semisimple algebraic \mathbb{Q} -group with no \mathbb{R} -anisotropic \mathbb{Q} -simple factors,
- $\mathbf{G}(\mathbb{Z}) \subset \mathbf{G}(\mathbb{Q})$ is the lattice stabilizer $G_{\rho,L}$ for some choice of ρ and L ,
- π is projection $\pi : \mathbf{G}(\mathbb{R})^\circ \rightarrow \overline{G}$ with compact kernel,
- $\Gamma \subset \overline{G}$ is a subgroup commensurable up to \overline{G} -automorphism to $\pi(\mathbf{G}(\mathbb{Z}) \cap \mathbf{G}(\mathbb{R})^\circ)$,
- $\varphi \in \text{Aut}(\overline{G})$ is such that $\pi(\mathbf{G}(\mathbb{Z}) \cap \mathbf{G}(\mathbb{R})^\circ)$ and $\varphi(\Gamma)$ are commensurable,
- $K \subset \mathbf{G}(\mathbb{R})$ is a maximal compact subgroup containing $\pi^{-1}(\varphi(\overline{K}))$.

An **arithmetic locally symmetric space (of noncompact type)** is a space M of the form $\Gamma \backslash \overline{G} / \overline{K}$. When Γ is torsion-free, M is a Riemannian manifold, and since every Γ has a finite index torsion-free subgroup [34], M is always a good Riemannian orbifold in the sense of Thurston [37, Chp. 13].

In this paper, we primarily study totally geodesic subspaces of arithmetic locally symmetric spaces. As we show in Theorem 3.2, totally geodesic subspaces inherit arithmeticity from its ambient space.

Theorem 3.2. *Let M be an arithmetic locally symmetric space and let $N \subset M$ be a nonflat, finite volume, totally geodesic subspace. Then N is arithmetic.*

Proof. By Lemma 2.2, there exists a connected, semisimple Lie subgroup $\overline{H} \subset \overline{G}$ with no compact factors such that $\tilde{N} := \overline{H} / (\overline{K} \cap \overline{H})$ is the universal cover of N and $\overline{\Lambda} \backslash \overline{H} / (\overline{K} \cap \overline{H})$ is commensurable to N where $\overline{\Lambda} := \Gamma \cap \overline{H}$ is a lattice in \overline{H} . Let H denote the connected component of the intersection of $\pi^{-1}(\varphi^{-1}(\overline{H}))$ with the noncompact factors of $\mathbf{G}(\mathbb{R})$. (This group can also be viewed as the unique connected Lie subgroup of $\mathbf{G}(\mathbb{R})$ with Lie algebra $\text{Lie}(\varphi^{-1}(\overline{H}))$.) It follows that $N' := \Lambda \backslash H / (K^\circ \cap H)$, where $\Lambda := \mathbf{G}(\mathbb{Z}) \cap H$, is commensurable with N . Arithmeticity is an invariant of commensurability class so it suffices to show the arithmeticity of N' . The result then follows by Lemma 3.3 below. \square

Lemma 3.3. *Let*

- (i) \mathbf{G} be an semisimple \mathbb{Q} -group,
- (ii) $H \subset \mathbf{G}(\mathbb{R})$ be a connected semisimple Lie subgroup with no compact factors, and
- (iii) $\Lambda \subset \mathbf{G}(\mathbb{Z})$ be a subgroup which is also a lattice in H .

Then $H = \mathbf{H}(\mathbb{R})^\circ$ where $\mathbf{H} \subset \mathbf{G}$ is a semisimple \mathbb{Q} -subgroup and $\Lambda \subset \mathbf{H}(\mathbb{Q})$ is arithmetic.

Proof. Since H is a semisimple Lie group sitting inside the real points of a linear group, H is the connected component of the real points of some semisimple \mathbb{R} -subgroup $\mathbf{H} \subset \mathbf{G}$. By Borel's Density Theorem [4] Λ is Zariski dense in \mathbf{H} . The Zariski closure of an abstract subgroup sitting inside the \mathbb{Q} -points of a group is also a \mathbb{Q} -group [2, Chp 1 Prop 1.3(b)]. Hence \mathbf{H} is defined over \mathbb{Q} . Now let $V := \text{Lie}(\mathbf{G})$ and $W := \text{Lie}(\mathbf{H})$. The adjoint representation $\text{Ad} : \mathbf{G} \rightarrow \mathbf{GL}(V)$ is defined over \mathbb{Q} . There exists a lattice $L \subset V$ which Γ stabilizes [3, Prop 7.12]. Since Λ stabilizes W , it stabilizes $L \cap W$ and hence Λ is an arithmetic subgroup of H . \square

If \mathbf{G}_1 and \mathbf{G}_2 are absolutely simple algebraic groups over number fields k_1 and k_2 , respectively, by [30, Prop 2.5], they give rise to commensurable arithmetic groups if and only if there is a field isomorphism $\tau : k_1 \rightarrow k_2$ such that \mathbf{G}_2 and $\mathbf{G}_1 \times_\tau \text{spec } k_2$ are isomorphic as k_2 -groups. We now give the following slight generalization of [30, Prop 2.5] that is useful when looking for totally geodesic subspaces.

Proposition 3.4. *Let M_1 and M_2 be arithmetic locally symmetric spaces arising from semisimple \mathbb{Q} -groups \mathbf{G}_1 and \mathbf{G}_2 respectively. Then M_1 and M_2 are commensurable if and only if \mathbf{G}_1 and \mathbf{G}_2 are \mathbb{Q} -isogenous.*

Proof. First suppose \mathbf{G}_1 and \mathbf{G}_2 are \mathbb{Q} -isogenous. Then $\mathrm{Ad}_{\mathbf{G}_1}(\mathbf{G}_1)$ and $\mathrm{Ad}_{\mathbf{G}_2}(\mathbf{G}_2)$ are \mathbb{Q} -isomorphic via a \mathbb{Q} -isomorphism ψ . Since M_i is commensurable with $\mathrm{Ad}_{\mathbf{G}_i}(\mathbf{G}_i(\mathbb{Z})) \backslash \mathrm{Ad}_{\mathbf{G}_i}(\mathbf{G}_i(\mathbb{R})) / \mathrm{Ad}_{\mathbf{G}_i}(K_i)$. The result then immediately follows from the fact that $\psi(\mathrm{Ad}_{\mathbf{G}_1}(\mathbf{G}_1(\mathbb{Z})))$ and $\mathrm{Ad}_{\mathbf{G}_2}(\mathbf{G}_2(\mathbb{Z}))$ are commensurable [3, Cor 7.13(2)].

Now suppose M_1 and M_2 are commensurable. By assumption, there exists a connected adjoint semisimple Lie group with no compact factors, \overline{G} , and two arithmetic lattices $\Gamma_1, \Gamma_2 \subset \overline{G}$ which are commensurable up to \overline{G} -automorphism, say ψ , such that $M_1 = \Gamma_1 \backslash \overline{G} / K$ and $M_2 = \Gamma_2 \backslash \overline{G} / \psi(K)$ where K is a maximal compact subgroup. Replacing \mathbf{G}_i with $\mathrm{Ad}_{\mathbf{G}_i}(\mathbf{G}_i)$, the result then follows from Lemma 3.5 below. \square

Lemma 3.5. *Let \overline{G} be a connected adjoint semisimple Lie group with no compact factors. Let $\Gamma_1, \Gamma_2 \subset \overline{G}$ be arithmetic lattices which are commensurable up to \overline{G} -automorphism. Let \mathbf{G}_1 and \mathbf{G}_2 be the connected adjoint semisimple \mathbb{Q} -groups with no \mathbb{R} -anisotropic \mathbb{Q} -simple factors giving rise to Γ_1 and Γ_2 respectively. Then \mathbf{G}_1 and \mathbf{G}_2 are \mathbb{Q} -isomorphic.*

Proof. Let ψ be an analytic automorphism of \overline{G} for which Γ_1 and $\psi(\Gamma_2)$ are commensurable. Let $\mathbf{H}_i \subset \mathbf{G}_i$ be the product of the connected \mathbb{R} -simple \mathbb{R} -isotropic components of \mathbf{G}_i . Then $\pi_i|_{\mathbf{H}_i(\mathbb{R})^\circ} : \mathbf{H}_i(\mathbb{R})^\circ \rightarrow \overline{G}$ is an isomorphism. Picking sufficiently small finite index $\Gamma'_i \subset \Gamma_i$ which are isomorphic via ψ , we may identify $\varphi_i^{-1}(\Gamma_i)$ with a finite index subgroup $\Lambda_i := \pi_i|_{\mathbf{H}_i(\mathbb{R})^\circ}^{-1}(\varphi_i^{-1}(\Gamma_i)) \subset \mathbf{H}_i(\mathbb{R})^\circ \cap \mathbf{G}_i(\mathbb{Q})$.

Since π_i induces an \mathbb{R} -rational isomorphism between \mathbf{H}_i and $\mathbf{Aut}(\mathrm{Lie}(\overline{G}) \otimes_{\mathbb{R}} \mathbb{C})$, and ψ induces an \mathbb{R} -rational automorphism on $\mathbf{Aut}(\mathrm{Lie}(\overline{G}) \otimes_{\mathbb{R}} \mathbb{C})$, it follows that there is an \mathbb{R} -rational isomorphism, which we also denote ψ , from \mathbf{H}_1 to \mathbf{H}_2 which sends Λ_1 to Λ_2 .

For each i , by [7, 6.21 (ii)], $\mathbf{G}_i \cong \prod_{j=1}^{r_i} R_{k_{i,j}/\mathbb{Q}} \mathbf{S}_{i,j}$ where \mathbf{S}_j is an absolutely simple group over a number field $k_{i,j}$. Then $\Lambda_{i,j} := \Lambda_i \cap (R_{k_{i,j}/\mathbb{Q}}(\mathbf{S}_{i,j}))(\mathbb{Q})$ is an arithmetic group in $(R_{k_{i,j}/\mathbb{Q}}(\mathbf{S}_{i,j}))(\mathbb{Q}) = \mathbf{S}_{i,j}(k_{i,j})$ [6, 6.11]. Borel's Density Theorem [5] implies that $\Lambda_{i,j}$ is Zariski dense in $\mathbf{S}_{i,j}$. Since each $\Lambda_{i,j}$ is a normal subgroup of Λ_i and an irreducible lattice in $(R_{k_{i,j}/\mathbb{Q}}(\mathbf{S}_{i,j}))(\mathbb{R})$, the isomorphism ψ must send each $\Lambda_{1,j}$ to some $\Lambda_{2,j'}$, from which we conclude $r_1 = r_2 := r$ and ψ induces a permutation also denoted $\psi \in S_r$. Our assumption on \mathbb{Q} -simple factors implies that each $R_{k_{i,j}/\mathbb{Q}} \mathbf{S}_{i,j}$ contains an \mathbb{R} -simple \mathbb{R} -isotropic factor. Since ψ sends \mathbb{R} -isotropic \mathbb{R} -simple factors of $R_{k_{1,j}/\mathbb{Q}} \mathbf{S}_{1,j}$ to \mathbb{R} -isotropic \mathbb{R} -simple factors of $R_{k_{2,\psi(j)}/\mathbb{Q}}(\mathbf{S}_{2,\psi(j)})$, we conclude $\mathbf{S}_{2,j}$ and $\mathbf{S}_{2,\psi(j)}$ have the same Killing–Cartan type. Let $\mathbf{H}_{i,j}$ be a fixed \mathbb{R} -simple \mathbb{R} -isotropic component of $R_{k_{i,j}/\mathbb{Q}} \mathbf{S}_{i,j}$. Then ψ induces an F -isomorphism between $\mathbf{S}_{1,j}$ and $\mathbf{S}_{2,\psi(j)}$, where $F = \mathbb{R}$ when $\mathbf{H}_{1,j}$ is absolutely simple, and $F = \mathbb{C}$ otherwise. Furthermore, this isomorphism sends $\Lambda_{1,j}$ to $\Lambda_{2,\psi(j)}$, hence by [30, Prop 2.5], $k_{1,j}$ and $k_{2,\psi(j)}$ are isomorphic and, letting k_j denote this isomorphism class (and changing the base of these groups), $\mathbf{S}_{1,j}$ and $\mathbf{S}_{2,\psi(j)}$ are k_j -isomorphic. The conclusion follows. \square

4. ARITHMETIC LOCALLY SYMMETRIC SPACES ARISING FROM QUADRATIC FORMS

In this section we discuss the theory of quadratic forms and the results we need to construct and analyze arithmetic locally symmetric spaces coming from quadratic forms. For a complete treatment of the classical theory of quadratic forms over local and global fields, we refer the reader to [28], [33], and [17].

Recall F is a field that is not of characteristic 2. In what follows, (V, q) will denote a quadratic space over F where V is a finite dimensional vector space over F and q is a quadratic form on V . When it will not cause confusion, we will omit V and simply refer to the quadratic form q . We

shall say q is a quadratic form over F , or more succinctly, q is a quadratic F -form. If E/F is a field extension then (V, q) determines a quadratic space (V_E, q_E) over E by extending scalars (i.e., where $V_E := V \otimes_F E$ and q_E is the extension of q to V_E). When it will not cause confusion, we will sometimes denote the extended form by the symbol q as well. Every quadratic space (V, q) determines an algebraic F -group, $\mathbf{SO}(V, q)$ whose E points are given by

$$\mathbf{SO}(V, q)(E) = \{T \in \mathbf{SL}(V_E) \mid q_E(Tv) = q_E(v) \text{ for all } v \in V_E\}.$$

Definition 4.1. Let (V_1, q_1) and (V_2, q_2) be quadratic spaces over F . Then q_1 and q_2 are

- (i) **isometric** if there some F -linear isomorphism $T : V_1 \rightarrow V_2$ such that $q_2(Tv) = q_1(v)$ for all $v \in V_1$.
- (ii) **similar** if there exists some $a \in F^\times$ such that q_1 and aq_2 are isometric.
- (iii) **isogroupic** if $\mathbf{SO}(q_1)$ and $\mathbf{SO}(q_2)$ are isomorphic as algebraic F -groups.

The first two definitions are standard, while the third we introduce in the paper. It is not hard to see that each of these determine an equivalence relation among quadratic F -forms. Furthermore, the following lemma begins to shows how they are related.

Lemma 4.2.

- (i) *Isometric forms are isogroupic.*
- (ii) *Similar forms are isogroupic.*

Proof.

(1) Let (V_1, q_1) and (V_2, q_2) be isometric forms. By assumption there exists an F -linear isomorphism $T : V_1 \rightarrow V_2$ preserving the forms. Then T induces an F -isomorphism $T_* : \mathbf{SL}(V_1) \rightarrow \mathbf{SL}(V_2)$ via $g \mapsto TgT^{-1}$. Upon restricting to $\mathbf{SO}(V_1, q_1)$, for any $v \in V_2$, we have

$$q_2(T_*(g)v) = q_2((TgT^{-1})v) = q_2(T(g(T^{-1}v))) = q_1(g(T^{-1}v)) = q_1(T^{-1}v) = q_2(v).$$

Hence $T_*(\mathbf{SO}(V_1, q_1)) \subset \mathbf{SO}(V_2, q_2)$ and by symmetry of argument it follows they are F -isomorphic.

(2) Let (V_1, q_1) and (V_2, q_2) be similar forms. By assumption there exists $a \in F^\times$ such that aq_1 and q_2 are isometric. By part (1), it suffices to show that aq_1 and q_1 are isogroupic. Pick $g \in \mathbf{SO}(V_1, q_1)$ and $v \in V_1$, then

$$(aq_1)(gv) = a(q_1(gv)) = a(q_1(v)) = (aq_1)(v).$$

Therefore $g \in \mathbf{SO}(V_1, aq_1)$, and by symmetry of argument, $\mathbf{SO}(V_1, q_1) = \mathbf{SO}(V_1, aq_1)$. The result follows. \square

In general there are many isometry classes in a given isogroupy class. If $\mathbf{G} := \mathbf{SO}(q)$, then any q' in the isogroupy class of q shall be said to **represent** \mathbf{G} .

A quadratic form r is a **subform** of a quadratic form q if there is some third form t such that $r \oplus t$ is isometric to q . We say a symmetric bilinear form b is **nondegenerate** when $b(v, w) = 0$ for all $w \in V$ implies that $v = 0$. A quadratic form corresponding to a nondegenerate symmetric bilinear form is said to be **regular**. In this paper, all quadratic forms will be assumed to be regular. The **dimension** of q , denoted $\dim q$, is the dimension of its associated vector space. When possible, we shall reserve the symbol m to denote the dimension of q . Upon choosing a basis, every quadratic form may be represented by an $m \times m$ matrix. The **determinant** of q , denoted $\det q$, is the determinant of some $Q \in \mathbf{GL}_m(F)$ representing q . Note however that since this should be independent of the choice of basis and $\det({}^tTQT) = \det Q(\det T)^2$, the determinant is only well defined up to square class of F , and hence we view $\det q \in F^\times / (F^\times)^2$. Though the determinant is a square class, we will often omit the $(F^\times)^2$ and write $\det q = a$ as opposed to $\det q = a(F^\times)^2$, where $a \in F^\times$. A common renormalization of the determinant is the **discriminant**, denoted $\text{disc}(q)$,

where $\text{disc}(q) = (-1)^{\dim(q)(\dim(q)-1)/2} \det(q)$. It contains the same information as the determinant if one knows the dimension, but often results in simpler expressions.

For $a, b \in F^\times$, the **Hilbert symbol** $\left(\frac{a, b}{F}\right) = (a, b)_F$ denotes the isomorphism class of the quaternion algebra generated by symbols i and j where $i^2 = a, j^2 = b$, and $ij = -ji$. When the field F is understood, we simply write (a, b) . The Hilbert symbol satisfies the following four properties that will be used frequently in this paper:

- (H1) Defined up to square class: $(a, bc^2) = (a, b)$,
- (H2) Symmetry: $(a, b) = (b, a)$,
- (H3) Multiplicativity: $(a_1 a_2, b) = (a_1, b)(a_2, b)$,
- (H4) Nondegeneracy: For $a \in F^\times$ not a square, there exists a $b \in F^\times$ such that $(a, b) \neq 1$.

Given a diagonal representation $\langle a_1, a_2, \dots, a_m \rangle$ of q [17, I.2.4], the **Hasse invariant** $c(q)$ is defined to be

$$(4.1) \quad c(q) := \begin{cases} \prod_{i < j} (a_i, a_j) & \text{if } m \geq 2, \text{ and} \\ 1 & \text{if } m = 1. \end{cases}$$

While this definition is common [8, 17], some authors use different normalizations of this invariant. In [28], the Hasse invariant is defined to be $c_{OM}(q) := \prod_{i \leq j} (a_i, a_j)$, and in [1], the Hasse invariant ϵ_{HW} is normalized so that the split quadratic form h always has $\epsilon_{HW}(h) = 1$. It follows that

$$(4.2) \quad c_{OM}(q) = c(q)(-1, \det q),$$

and, by our computations in Section 5, which are summarized in Table 5.1, for $m \geq 4$,

$$(4.3) \quad \epsilon_{HW}(q) = \begin{cases} c(q)(-1, -1)^{\frac{n(n-1)}{2}} & \text{if } m = 2n, \\ c(q)(-1, -1)^{\frac{n(n-3)}{2}}(-1, \det q)^n & \text{if } m = 2n + 1. \end{cases}$$

The Hasse invariant satisfies the following useful product formula:

$$(4.4) \quad c(q_1 \oplus q_2) = c(q_1)c(q_2)(\det q_1, \det q_2).$$

While the Hasse invariant is a well defined invariant of the isometry class of q [17, V.3.8], it is not an invariant of the similarity class of q . The relationship between $c(q)$ and $c(\lambda q)$, $\lambda \in F^\times$, is given by the following lemma.

Lemma 4.3. *Let F be a field of characteristic not 2, let q be a quadratic form over F of dimension m , and let $\lambda \in F^\times$. Then*

$$c(\lambda q) = \left(\lambda, (-1)^{\frac{m(m-1)}{2}} (\det q)^{m-1} \right) c(q).$$

In particular this reduces to

$$(4.5) \quad c(\lambda q) = \begin{cases} (\lambda, \text{disc}(q)) c(q) & \text{when } m \text{ is even,} \\ \left(\lambda, (-1)^{\frac{m-1}{2}} \right) c(q) & \text{when } m \text{ is odd.} \end{cases}$$

Proof. A direct computation gives:

$$\begin{aligned} c(\lambda q) &= \prod_{i < j} (\lambda a_i, \lambda a_j) \\ &= \prod_{i < j} (\lambda, \lambda)(\lambda, a_i)(\lambda, a_j)(a_i, a_j) \\ &= (\lambda, -1)^{\frac{m(m-1)}{2}} (\lambda, \det q)^{m-1} c(q) \\ &= \left(\lambda, (-1)^{\frac{m(m-1)}{2}} (\det q)^{m-1} \right) c(q). \end{aligned}$$

The reduction when m is even and odd immediately follows. \square

The extent to which the Hasse invariant varies within an isogroupy class will be explored in Section 5. In general the Hasse invariant is difficult to compute, however, when F is a nonarchimedean local field or \mathbb{R} , then $c(q)$ can only take values ± 1 , and over \mathbb{C} , $c(q)$ is identically 1.

Every isometry class of quadratic forms over \mathbb{R} can be diagonally represented with the first m_+ terms positive and the remaining $m_- := m - m_+$ terms negative. The **signature** of q is the pair $\text{sgn}(q) := (m_+, m_-)$. Some authors define to the signature of q to be the number $s = m_+ - m_-$. Observe that the two pairs (m, s) and (m_+, m_-) contain equivalent information. The signature is an invariant of the isometry class of q , and the unordered pair $\{m_+, m_-\}$ is an invariant of the similarity class of q .

These invariants determine the isometry classes of quadratic forms over local and global fields. For the reader's convenience, we state the uniqueness and existence theorems for quadratic forms over local and global fields. These will be essential in our analysis in later sections.

Theorem 4.4 (Local Uniqueness). *Let F be \mathbb{C} , \mathbb{R} , or a finite extension of \mathbb{Q}_p that we denote L , and let q and q' be quadratic F -forms. Then q and q' are isometric if and only if*

- (i) *When $F = \mathbb{C}$, $\dim q = \dim q'$.*
- (ii) *When $F = \mathbb{R}$, $\dim q = \dim q'$ and $\text{sgn}(q) = \text{sgn}(q')$.*
- (iii) *When $F = L$, $\dim q = \dim q'$, $\det(q) = \det(q')$, and $c(q) = c(q')$.*

Theorem 4.5 (Local Existence).

- (i) *For each $m \in \mathbb{Z}_{\geq 1}$, there exists a quadratic \mathbb{C} -form q such that*

$$\dim q = m.$$
- (ii) *For each pair $(m_+, m_-) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$, there exists a quadratic \mathbb{R} -form q such that*

$$\dim q = m := m_+ + m_- \quad \text{and} \quad \text{sgn}(q) = (m_+, m_-).$$
- (iii) *For each triple $(m, d, c) \in \mathbb{Z}_{\geq 1} \times L^\times / (L^\times)^2 \times \{\pm 1\}$, there exists a quadratic L -form q such that*

$$\dim q = m, \quad \det q = d \quad \text{and} \quad c(q) = c,$$
- (*) *with the exception that $c = 1$ when either $m = 1$ or $m = 2$ and $d = -1$.*

While the exceptional restrictions (*) on the Hasse invariant in dimensions $m = 1$ and $m = 2$ may seem inconsequential, they will play an integral role in the construction of subforms later in the paper. For more on the above results over \mathbb{R} and L , we refer the reader to [36] and [28, VI.63:23], respectively.

Theorem 4.6 (Local-to-Global Uniqueness). [28, VI.66:4] *Let k be a number field and q and q' be quadratic k -forms. Then $q \cong q'$ if and only if $q \otimes k_v \cong q' \otimes k_v$ for all $v \in V_k$.*

Theorem 4.7 (Local-to-Global Existence). [28, VII.72:1] *Let k be a number field and let*

- $m \in \mathbb{Z}_{\geq 1}$,
- $d \in k^\times / (k^\times)^2$, and
- $S \subset V_k$ be a finite subset of even cardinality.

For each family $\{q_v\}_{v \in V_k}$ where q_v is a quadratic form over k_v satisfying

- $\dim q_v = m$,
- $\det q_v = d$, and
- $c_v(q_v) = -1$ if and only if $v \in S$,

there exists a quadratic form q over k such that $q \otimes k_v = q_v$ for all $v \in V_k$.

Quadratic forms are used to construct irreducible arithmetic lattices of semisimple Lie groups of the form

$$G = \prod_{i=1}^r \mathbf{SO}(p_i, m - p_i) \times (\mathbf{SO}_m(\mathbb{C}))^s.$$

In the literature, these lattices have been called: “standard” [19], “lattice of the simplest type” [41], and, “coming from quadratic forms.” We shall use the terminology “standard” when convenient, or otherwise we shall say explicitly “coming from quadratic forms.”

Construction 4.8. Fix the following notation:

- (i) k is a number field with infinite places V_k^∞ ;
- (ii) (V, q) is an m -dimensional quadratic k -space, $m \geq 3$;
- (iii) $\mathbf{G} := \mathbf{SO}(V, q)$ is the absolutely almost simple k -group defined by (V, q) and $SO(q) := \mathbf{G}(k)$;
- (iv) For each $v \in V_k^\infty$, $V_{k_v} := V \otimes_k k_v$, $q_v := q \otimes k_v$, and \mathbf{G}_v is the algebraic k_v -group $\mathbf{SO}(V_{k_v}, q_v)$,
 - If v is real, then $\mathbf{G}_v(k_v) \cong \mathbf{SO}(m_+^{(v)}, m_-^{(v)})$,
 - If v is complex, then $\mathbf{G}_v(k_v) \cong \mathbf{SO}_m(\mathbb{C})$,
 - r is the number of real places where q is isotropic,
 - s is the number of complex places, and
 - $p_i := m_+^{(v_i)}$ where $\{v_1, \dots, v_r\}$ is the set of real places where q is isotropic;
- (v) $\mathbf{G}' := R_{k/\mathbb{Q}} \mathbf{G}$ is the semisimple \mathbb{Q} -group formed by restriction of scalars. Then $\mathbf{G}'(\mathbb{R}) = \prod \mathbf{G}_v(k_v)$ is a semisimple Lie group that has compact factors at precisely the real places where q is anisotropic. There is an isomorphism $\mathbf{G}(k) \cong \mathbf{G}'(\mathbb{Q})$ and diagonal embedding $SO(q) \rightarrow \mathbf{G}'(\mathbb{R})$;
- (vi) G is the projection of $\mathbf{G}'(\mathbb{R})$ onto its noncompact factors and denote the projection map by $\pi : \mathbf{G}'(\mathbb{R}) \rightarrow G$. Observe that G is a semisimple Lie group with no compact factors and is \mathbb{R} -simple when $r + s = 1$;
- (vii) $L \subset V$ is an \mathcal{O}_k -lattice and $G_L := \{T \in \mathbf{G}(k) \mid T(L) \subset L\}$. Then G_L sits as a discrete arithmetic subgroup of the semisimple Lie group $\mathbf{G}'(\mathbb{R})$;
- (viii) $\Gamma \subset G$ is commensurable up to G -automorphism with $\pi(G_L)$. Then Γ is said to be a **standard arithmetic lattice of G** . Figure 4.1 below summarizes this construction.

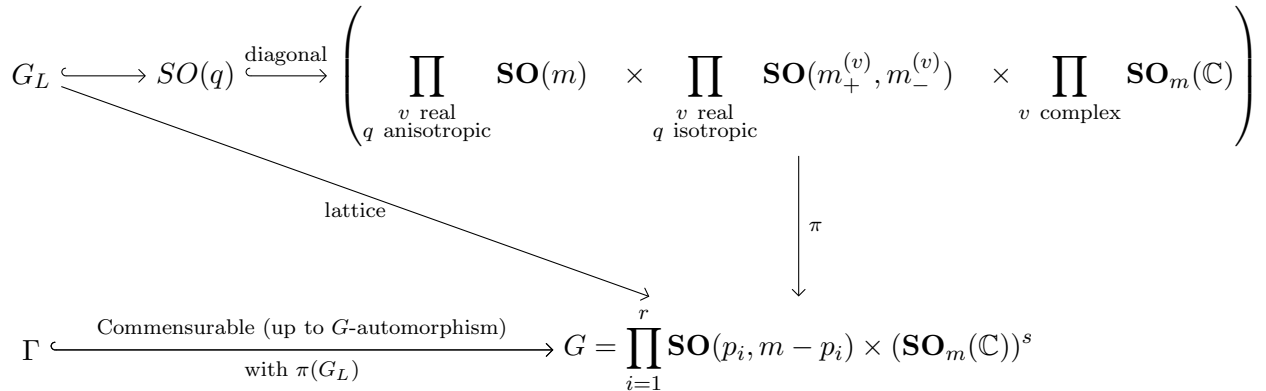


FIGURE 4.1. Construction of Standard arithmetic lattices in G .

- (ix) $K \subset G$ is a maximal compact subgroup and $M_\Gamma := \Gamma \backslash G / K$.
 - (a) M_Γ is an **arithmetic locally symmetric space coming from a quadratic form**, (or a **standard arithmetic locally symmetric space of type B_n or D_n**), and
 - (b) $k(M_\Gamma) := k$ is the **field of definition** of M_Γ .

When M_Γ is simple (i.e. $r = 1$ and $m \neq 4$), [30, Lemma 2.6] implies that $k(M_\Gamma)$ coincides with the minimal field of definition of Γ in the sense of Vinberg [40]. A choice of another \mathcal{O}_k -lattice $L' \subset V$ and Γ' commensurable up to G -automorphism with $\pi(G_{L'})$ will produce a space $M_{\Gamma'}$ that is commensurable with M_Γ . Hence choosing the pair (k, q) determines a commensurability class which we will sometimes denote by M_q . Conversely, if the pairs (k_1, q_1) and (k_2, q_2) yield commensurable spaces, by [30, Prop. 2.5], there is a field isomorphism $\tau : k_1 \rightarrow k_2$ such that $\mathbf{SO}(q_2)$ and $\mathbf{SO}(q_1 \otimes_\tau k_2)$ are k_2 -isomorphic groups.

Definition 4.9. If (V_1, q_1) is a quadratic space over k_1 and $\tau : k_1 \rightarrow k_2$ is a field isomorphism, then $(V_2, q_2) := (V_1 \otimes_\tau k_2, q_1 \otimes_\tau k_2)$ is a quadratic space over k_2 . We call such base change a **twist** by τ . If such a τ is implicit, we just say (V_2, q_2) is a **twist** of (V_1, q_1) .

By our remarks above, a twist of (V, q) and (V, q) yield commensurable arithmetic lattices. In particular, a commensurability class of standard arithmetic lattices of type B_n or D_n is uniquely determined by a twist class of an isogroupy class of quadratic k -forms. Given a number field k with a fixed real (resp. complex) place v_0 , and a quadric k -form q isotropic at a unique real (resp. complex) place, there is always a twist q' of q that is isotropic at v_0 .

Definition 4.10. We call (k, q) an **admissible hyperbolic pair** if M_q is a commensurability class of hyperbolic orbifolds, that is to say,

- (i) k is totally real (i.e., no $\mathbf{SO}_m(\mathbb{C})$ terms),
- (ii) q is anisotropic at all but one real place, v_0 (i.e., only one $\mathbf{SO}(p_i, m - p_i)$ term),
- (iii) $q \otimes k_{v_0}$ has signature $(m - 1, 1)$ or $(1, m - 1)$. (i.e., $G \cong \mathbf{SO}(m - 1, 1)$).

When $m \geq 3$ is odd, all irreducible arithmetic lattices of G arise from Construction 4.8 ([38] [19, §3]). When $m > 3$, $m \neq 8$, is even, all other are irreducible arithmetic lattices of G come from skew Hermitian forms over quaternion division algebras over number fields (See Construction 10.1). When $m = 8$, in addition to lattices coming from skew Hermitian forms, there are also lattices which come from triality.

Construction 4.11. Let (W, r) be a quadratic k -subspace of (V, q) . Then $\mathbf{H} = \mathbf{SO}(W, r)$ is an absolutely almost simple k -subgroup of \mathbf{G} . Let $\mathbf{H}' := R_{k/\mathbb{Q}}\mathbf{H}$. Then \mathbf{H}' is a semisimple \mathbb{Q} -subgroup of \mathbf{G}' . It follows that $L \cap W$ is an \mathcal{O}_k -lattice of W , hence $G_L \cap \mathbf{H}'(\mathbb{R})$ is an arithmetic subgroup of $\mathbf{H}'(\mathbb{R})$. Let H be the image of $\mathbf{H}'(\mathbb{R})$ under the projection map π onto the noncompact factors of $\mathbf{G}'(\mathbb{R})$. Then $\pi(G_L \cap \mathbf{H}'(\mathbb{R}))$ is an arithmetic subgroup of H . Note that H may be trivial. It follows that $N_{\Gamma \cap H} := (\Gamma \cap H) \backslash H / (H \cap K)$ is commensurable to a totally geodesic subspace of M_Γ . We denote this commensurability class N_r . In what follows, we shall call such totally geodesic subspaces **subform subspaces**. Observe that for a subform subspace $N \subset M$, $k(N) = k(M)$. Furthermore, if $\dim r \geq 2$ and r is isotropic at a real place of k , then N_r is a commensurability class of nontrivial, nonflat, finite volume, locally symmetric spaces of noncompact type.

5. THE INDEX OF ISOMETRY GROUPS OF QUADRATIC FORMS

Let \mathbf{G} be an absolutely almost simple algebraic k -group. We will use the conventions of [38] and denote the **Tits index** of \mathbf{G} by ${}^g X_{n,r}^{(d)}$ where:

- (i) X_n is the Killing–Cartan type of $\mathbf{G} \otimes \bar{k}$,
- (ii) n is the \bar{k} -rank of \mathbf{G}
- (iii) g is the order of the image of the $*$ -action map
- (iv) r is the k -rank of \mathbf{G} , and
- (v) d is the degree of a division algebra associated with \mathbf{G} .

When $\mathbf{G} = \mathbf{SO}(q)$, $g = 1$ or $g = 2$ (depending on whether \mathbf{G} is an inner or outer form), and $d = 1$. The Tits index encodes a large amount of information about a semisimple algebraic group's isogeny

class [38, Theorem 2.7.1]. We refer the reader to [38] for more information. One of the goals of this section is to relate the local index of $\mathbf{SO}(q)$ to the local invariants of q (see Table 5.1).

For the reader's convenience, we state two basic results relating a quadratic form's invariants to whether or not it is isotropic.

Lemma 5.1. [8, Chp. 4 Lem 2.5 & Lem 2.6] *Let L be a nonarchimedean local field.*

- (i) *Let q' be a 3-dimensional quadratic form over L . Then q' is isotropic if and only if $c(q') = (-1, -\det q')$.*
- (ii) *Let q' be a 4-dimensional quadratic form over L . Then q' is anisotropic if and only if $\text{disc}(q') = 1$ and $c(q') = -(-1, -1)$.*

Though the proofs in [8] are explicitly written with $k = \mathbb{Q}$, they are generalizable to an arbitrary number field.

Proposition 5.2. *Let k be a number field, $v \in V_k$ be a finite place, and q be a $(2n+1)$ -dimensional quadratic form over k_v . Then the local index of the k_v -group $\mathbf{SO}(q)$ is $B_{n,n}$ if and only if*

$$(5.1) \quad c(q) = (-1, -1)^{\frac{n(n-3)}{2}} (-1, \det q)^n.$$

Proof. We will show that the following statements are equivalent:

- (i) $\mathbf{SO}(q)$ is of type $B_{n,n}$.
- (ii) $q \cong \langle 1, -1 \rangle^{n-1} \oplus q'$ where q' is an isotropic 3-dimensional form.
- (iii) $q \cong \langle 1, -1 \rangle^{n-1} \oplus q'$ where $c(q') = (-1, -\det q')$.
- (iv) $c(q) = (-1, -1)^{\frac{n(n-3)}{2}} (-1, \det q)^n$.

First (1) is equivalent to (2) by the classification of algebraic k_v -groups in [38]. Next, (2) is equivalent to (3) by Proposition 5.1 (1). Lastly (3) is equivalent to (4) by the following computation:

$$\begin{aligned} c(q) &= c(\langle 1, -1 \rangle^{n-1} \oplus q') \\ &= c(\langle 1, -1 \rangle^{n-1}) c(q') ((-1)^{n-1}, \det q') \\ &= (-1, -1)^{\frac{(n-1)(n-2)}{2}} (-1, -\det q') (-1, \det q')^{n-1} \\ &= (-1, -1)^{\frac{(n-1)(n-2)}{2} + 1} (-1, \det q')^n \\ &= (-1, -1)^{\frac{(n^2 - 3n + 2 + 2)}{2}} (-1, (-1)^{n-1} \det q)^n \\ &= (-1, -1)^{\frac{n^2 - 3n}{2} + 2} (-1, \det q)^n \\ &= (-1, -1)^{\frac{n(n-3)}{2}} (-1, \det q)^n. \end{aligned}$$

□

Proposition 5.3. *Let k be a number field, $v \in V_k$ be a finite place, and q be a $(2n)$ -dimensional quadratic form over k_v . Then the local index of the k_v -group $\mathbf{SO}(q)$ is ${}^1D_{n,n-2}$ if and only if*

$$(5.2) \quad \text{disc } q = 1 \quad \text{and} \quad c(q) = -(-1, -1)^{\frac{n(n-1)}{2}}.$$

Proof. We will show that the following statements are equivalent:

- (i) $\mathbf{SO}(q)$ is of type ${}^1D_{n,n-2}$.
- (ii) $q = \langle 1, -1 \rangle^{n-2} \oplus q'$ where q' is an anisotropic 4-dimensional form.
- (iii) $q = \langle 1, -1 \rangle^{n-2} \oplus q'$ where $\text{disc } q' = 1$ and $c(q') = -(-1, -1)$.
- (iv) $\text{disc } q = 1$ and $c(q) = -(-1, -1)^{\frac{n(n-1)}{2}}$.

Type	Classical Invariants	Tits Index
$B_{n,n}$	$\dim(q) = 2n + 1$ $\det(q) = \text{anything}$ $c(q) = (-1, -1)^{\frac{n(n-3)}{2}} (-1, \det(q))^n$	
$B_{n,n-1}$	$\dim(q) = 2n + 1$ $\det(q) = \text{anything}$ $c(q) = -(-1, -1)^{\frac{n(n-3)}{2}} (-1, \det(q))^n$	
${}^1D_{n,n}^{(1)}$	$\dim(q) = 2n$ $\det(q) = (-1)^n$ (i.e. $\text{disc}(q) = 1$) $c(q) = (-1, -1)^{\frac{n(n-1)}{2}}$	
${}^1D_{n,n-2}^{(1)}$	$\dim(q) = 2n$ $\det(q) = (-1)^n$ (i.e. $\text{disc}(q) = 1$) $c(q) = -(-1, -1)^{\frac{n(n-1)}{2}}$	
${}^2D_{n,n-1}^{(1)}$	$\dim(q) = 2n$ $\det(q) \neq (-1)^n$ (i.e. $\text{disc}(q) \neq 1$) $c(q) = \text{anything}$	

Table 5.1: Dictionary between the classical invariants of q and the index of $\mathbf{SO}(q)$.

First (1) is equivalent to (2) by the classification of algebraic k -groups in [38]. Next, (2) is equivalent to (3) by Proposition 5.1 (2). Lastly (3) is equivalent to (4) by the following computations:

$$\begin{aligned}
\text{disc } q &= \text{disc}(\langle 1, -1 \rangle^{n-2} \oplus q') \\
&= \text{disc}(\langle 1, -1 \rangle^{n-2}) \text{disc } q' \\
&= 1.
\end{aligned}
\qquad
\begin{aligned}
c(q) &= c(\langle 1, -1 \rangle^{n-2} \oplus q') \\
&= c(\langle 1, -1 \rangle^{n-2}) c(q') ((-1)^{n-2}, \det q') \\
&= (-1, -1)^{\frac{(n-2)(n-3)}{2}} - (-1, -1) \\
&= -(-1, -1)^{\frac{(n-2)(n-3)}{2} + 1} \\
&= -(-1, -1)^{\frac{(n^2 - 5n - 6 + 2)}{2}} \\
&= -(-1, -1)^{\frac{(n^2 - n)}{2} - 2(n-1)} \\
&= -(-1, -1)^{\frac{n(n-1)}{2}}.
\end{aligned}$$

□

Proposition 5.4. *Let k be a number field, m be odd, and q and q' be m -dimensional quadratic form over k . Then q and q' are isogroupic if and only if they are similar.*

Proof. In Lemma 4.2, we showed similar forms are isogroupic. Now suppose q' represents $\mathbf{G} := \mathbf{SO}(q)$. Let $a \in k^\times / (k^\times)^2$ such that $\det q' = a \det q$. We shall show aq and q' are isometric. Note that aq also represents \mathbf{G} , and since m is odd, $\det(aq) = a \det q = \det q'$. We now look at the forms locally.

- (i) At each complex place $v \in V_k$, aq and q' have the same dimension, and hence are isometric by Theorem 4.4 (a).
- (ii) At each real place $v \in V_k$, since m is odd, the index of \mathbf{G} together with the determinant $\det q'$ uniquely determines the signature of $q' \otimes k_v$. Hence at each real place, $\text{sgn}(q') = \text{sgn}(aq)$. By Theorem 4.4 (b), aq and q' are isometric at each real place.
- (iii) At each finite place $v \in V_k$, since m is odd, equation (5.1) shows that the index of \mathbf{G} together with $\det q'$ uniquely determines $c(q')$. Hence at each finite place, $c(q') = c(aq)$. By Theorem 4.4 (c), aq and q' are isometric at each finite place.

By Theorem 4.6, aq and q' are isometric over k and the result follows. \square

In [13, 2.6], there is an analogous result for admissible hyperbolic pairs of any dimension. Their proof heavily uses hyperbolic geometry while our proof is algebraic in nature and applies to all odd dimensional forms.

Proposition 5.5. *Let k be a number field, q_1 and q_2 be $m = 2n + 1$ -dimensional quadratic forms over k , and $\mathbf{G}_i = \mathbf{SO}(q_i)$. Then \mathbf{G}_1 and \mathbf{G}_2 are k -isomorphic if and only if the groups $\mathbf{G}_1 \otimes k_v$ and $\mathbf{G}_2 \otimes k_v$ have the same local index for all $v \in V_k$.*

In particular, the k -isomorphism class of $\mathbf{G} := \mathbf{SO}(q)$ is determined by its index at all places.

Proof. If \mathbf{G}_1 and \mathbf{G}_2 are k -isomorphic, then $\mathbf{G}_1 \otimes k_v$ and $\mathbf{G}_2 \otimes k_v$ are k_v -isomorphic for all $v \in V_k$, and hence by the Tits Classification Theorem [38, Theorem 2.7.1], they have the same index at every place.

We now prove the other direction and suppose that $\mathbf{G}_1 \otimes k_v$ and $\mathbf{G}_2 \otimes k_v$ have the same index for all $v \in V_k$. We may replace q_2 with the similar form $\frac{\det q_1}{\det q_2} q_2$, and since m is odd, we may now assume $\det q_1 = \det q_2$. As we observed in the proof of the previous proposition, at local places the index and the determinant determine the isometry class of a representing form. Therefore $q_1 \otimes k_v$ and $q_2 \otimes k_v$ are isometric for all $v \in V_k$, and hence by Theorem 4.6, q_1 and q_2 are isometric. The result follows from Lemma 4.2. \square

6. FIELDS OF DEFINITION AND THE PROOF OF THEOREM A

Let \mathbf{G} be a semisimple algebraic group over \mathbb{C} and let $\Gamma \subset \mathbf{G}(\mathbb{C})$ be a Zariski-dense subgroup. A **field of definition** for Γ is a field $F \subset \mathbb{C}$ for which there exists an F -form \mathbf{G}' of \mathbf{G} and an isomorphism $\varphi : \mathbf{G} \rightarrow \mathbf{G}'$ defined over a finite extension of F such that $\varphi(\Gamma) \subset \mathbf{G}'(F)$ [22, 10.3.10]. Vinberg showed [40] that for Zariski-dense groups, there is a unique minimal field of definition

$$k_{\mathbf{G}}(\Gamma) := \mathbb{Q}(\text{Tr}(\text{Ad}_{\mathbf{G}}(\gamma)) \mid \gamma \in \Gamma),$$

where $\text{Ad}_{\mathbf{G}}$ is the adjoint representation of \mathbf{G} . Furthermore this is an invariant of the commensurability class. In general, the minimal field of definition of a Zariski-dense Γ need not coincide with the field that \mathbf{G} is defined over. Furthermore, the same abstract group can have different fields of definition depending on the ambient group. However, [30, Prop. 2.6] showed that for an absolutely almost simple group \mathbf{G} over a number field k , and $\Gamma \subset \mathbf{G}(k)$ arithmetic and Zariski-dense, the minimal field of definition of Γ coincides with the field of definition of the group (i.e. $k_{\mathbf{G}}(\Gamma) = k$). As such, the field of definition of Construction 4.8 coincides with the minimal field of definition in the sense of Vinberg [40] for all quadratic forms q of dimension $m > 2$ and $m \neq 4$ (since it is in this case and only in this case that $\mathbf{SO}(q)$ is not absolutely almost simple).

For an arbitrary totally geodesic subspace $N \subset M$, we do not expect to see a relationship between $k(N)$ and $k(M)$ as is demonstrated by the following examples.

Example 6.1. We show three methods of constructing of totally geodesic subspaces $N \subset M$ where both N and M come from quadratic forms and where each realizes a different relationship between $k(N)$ and $k(M)$.

- (i) Subforms produce $N \subset M$ such that $k(N) = k(M)$.
Let k be an arbitrary number field and let q be a quadratic form over k of dimension ≥ 4 . Let $r \subset q$ be a subform of dimension ≥ 3 . Then $\mathbf{SO}(r)$ naturally sits inside $\mathbf{SO}(q)$ as a k -subgroup. Then $k(N) = k = k(M)$.
- (ii) Extension of scalars produce $N \subset M$ such that $k(N) \subsetneq k(M)$.
Let k/\mathbb{Q} be a nontrivial finite extension and let q be a quadratic form over \mathbb{Q} of dimension ≥ 3 . Then $\mathbf{SO}(q)$ naturally sits as a \mathbb{Q} -subgroup in the diagonal of $R_{k/\mathbb{Q}}(\mathbf{SO}(q \otimes_{\mathbb{Q}} k))$. Then $k(N) = \mathbb{Q} \subsetneq k = k(M)$.
- (iii) Killing form produces $N \subset M$ such that $k(N) \supsetneq k(M)$.
Let k/\mathbb{Q} be a nontrivial finite extension, let q be a quadratic form over k of dimension ≥ 3 , let $\mathbf{H} = \mathbf{SO}(q)$, and $\mathbf{G} = \mathbf{SO}(\mathrm{Lie}(R_{k/\mathbb{Q}}(\mathbf{H})), \kappa)$ where κ is the Killing form on $\mathrm{Lie}(R_{k/\mathbb{Q}}(\mathbf{H}))$. Then, via the adjoint representation,

$$\mathbf{H}(k) = (R_{k/\mathbb{Q}}(\mathbf{H}))(\mathbb{Q}) \subset (\mathbf{SO}(\mathrm{Lie}(R_{k/\mathbb{Q}}(\mathbf{H})), \kappa))^{\circ}(\mathbb{Q}) \subset \mathbf{G}(\mathbb{Q}).$$

Then $k(N) = k \supsetneq \mathbb{Q} = k(M)$.

Observe that in the above examples, when $k(N) \neq k(M)$, the difference between $\dim N$ and $\dim M$ was quite large. As the next results show, if the dimensions of N and M are sufficiently close, there is a relationship between their fields of definition.

Lemma 6.2. *For $i = 1, 2$, let \mathbf{H}_i be semisimple k_i -groups such that \mathbf{H}_1 is absolutely almost simple and $R_{k_1/\mathbb{Q}}(\mathbf{H}_1)$ is \mathbb{Q} -isogenous to $R_{k_2/\mathbb{Q}}(\mathbf{H}_2)$.*

- (i) *Then k_2 is isomorphic to a subfield of k_1 .*
- (ii) *If $\dim \mathbf{H}_2 < 2 \dim \mathbf{H}_1$, then there is a field isomorphism $\tau : k_1 \rightarrow k_2$ and $\mathbf{H}_1 \times_{\kappa} \mathrm{spec} k_2$ and \mathbf{H}_2 are k_2 -isogenous.*

Proof.

(1) Replacing \mathbf{H}_i by their adjoint groups, we have $R_{k_1/\mathbb{Q}}(\mathbf{H}_1)$ and $R_{k_2/\mathbb{Q}}(\mathbf{H}_2)$ are \mathbb{Q} -isomorphic. Since \mathbf{H}_1 is absolutely simple, $R_{k_1/\mathbb{Q}}(\mathbf{H}_1)$ is \mathbb{Q} -simple and hence $R_{k_2/\mathbb{Q}}(\mathbf{H}_2)$ is \mathbb{Q} -simple. It follows that \mathbf{H}_2 must be k_2 -simple, and by Proposition 3.1 (1), there exists a field extension k'_1/k_2 and absolutely simple k'_1 -group \mathbf{H}'_1 such that $R_{k'_1/k_2}(\mathbf{H}'_1)$ and \mathbf{H}_2 are k_2 -isomorphic. It follows that $R_{k_1/\mathbb{Q}}(\mathbf{H}_1)$ and $R_{k'_1/k_2}(R_{k_2/\mathbb{Q}}(\mathbf{H}'_1)) \cong R_{k'_1/\mathbb{Q}}(\mathbf{H}'_1)$ are \mathbb{Q} -isomorphic. By Proposition 3.1 (2), there is a field isomorphism $\tau : k_1 \rightarrow k'_1$ and $\mathbf{H}_1 \times_{\kappa} \mathrm{spec} k_2$ and \mathbf{H}'_1 are k'_1 -isomorphic.

(2) Our initial assumptions imply that \mathbf{H}_2 is $\overline{\mathbb{Q}}$ -isomorphic to $\deg_{k_2}(k'_1)$ copies of \mathbf{H}_1 . The restriction on dimension implies that \mathbf{H}_2 has precisely one such simple factor. Hence $k'_1 = k_2$ and the result follows. \square

Proposition 6.3. *Let \mathbf{H}_1 be an absolutely almost simple k_1 -group and \mathbf{G} be absolutely almost simple k_2 -group, both of which are isotropic at precisely one infinite place, such that $\dim \mathbf{G} < 2 \dim \mathbf{H}_1$. Suppose $R_{k_1/\mathbb{Q}}(\mathbf{H}_1)$ is \mathbb{Q} -isogenous to a \mathbb{Q} -subgroup of $R_{k_2/\mathbb{Q}}(\mathbf{G})$. Then k_1 and k_2 are isomorphic.*

Proof. Replace \mathbf{H}_1 and \mathbf{G} by their adjoint groups and let v_1 (resp. v_2) denote the unique infinite place of k_1 (resp. k_2) where \mathbf{H}_1 (resp. \mathbf{G}) is isotropic. Then there is an injective \mathbb{Q} -rational map,

$$\varphi : R_{k_1/\mathbb{Q}}(\mathbf{H}_1) \rightarrow R_{k_2/\mathbb{Q}}(\mathbf{G}),$$

which induces an injective map of \mathbb{R} -simple Lie groups

$$\varphi : \mathbf{H}_1(k_{1,v_1}) \rightarrow \mathbf{G}(k_{2,v_2}).$$

Let \mathbf{H}_2 denote the Zariski-closure of $\varphi(\mathbf{H}_1(k))$ in $\mathbf{G}(k_{2,v_2})$. Since $\varphi(\mathbf{H}_1(k_1)) \subset \mathbf{G}(k_2)$, \mathbf{H}_2 is defined over k_2 . Observe that $R_{k_1/\mathbb{Q}}(\mathbf{H}_1)$ is \mathbb{Q} -isogenous to $R_{k_2/\mathbb{Q}}(\mathbf{H}_2)$ and by our assumption on dimension, $\dim \mathbf{H}_2 \leq \dim \mathbf{G} < 2 \dim \mathbf{H}_1$. Therefore by Lemma 6.2 (2), the result follows. \square

Observe that in the proof of Proposition 6.3, we use the fact that our groups are isotropic at precisely one infinite place to ensure that \mathbf{H}_2 is defined over k_2 , instead of a proper subfield, and that the dimension of \mathbf{H}_2 satisfies the bounds of Lemma 6.2 (2).

Proposition 6.4. *Let M_1 and M_2 be arithmetic locally symmetric spaces coming from quadratic forms q_1 and q_2 of dimension ≥ 4 over number fields k_1 and k_2 respectively. If $\mathbb{Q}TG(M_1) = \mathbb{Q}TG(M_2)$, then $\dim q_1 = \dim q_2$.*

Proof. We shall prove the contrapositive. If $\dim q_1 \neq \dim q_2$, then (potentially after relabeling), $\dim q_1 > \dim q_2$. Let $v_0 \in V_{k_1}$ be a real place where $q_1 \otimes k_{1,v_0}$ is isotropic. By deleting one entry in a diagonal representation of q_1 we have a $(\dim q_1 - 1)$ -dimensional form r that is isotropic at v_0 . Let $\mathbf{H} := \mathbf{SO}(r)$, H denote the noncompact factors of $R_{k_1/\mathbb{Q}}(\mathbf{H})$, and $\mathbf{G}_2 := \mathbf{SO}(q_2)$. Considering the dimensions of the simple factors of H , it follows that H cannot be \mathbb{R} -isogenous to a proper subgroup of $R_{k_2/\mathbb{Q}}(\mathbf{G}_2)$. Since $\dim r \geq 3$, r gives rise to a nonflat finite volume totally geodesic subspace N of M_1 that cannot be a proper totally geodesic subspace of M_2 . The result then follows. \square

Proof of Theorem A. Let q_1 and q_2 be quadratic forms over k_1 and k_2 , giving rise to M_1 and M_2 , respectively. Since $\mathbb{Q}TG(M_1) = \mathbb{Q}TG(M_2)$, Proposition 6.4 implies $\dim q_1 = \dim q_2 =: m$. Let r be an $(m-1)$ -dimensional quadratic k_1 -subform of q_1 that is isotropic at the real place where q_1 is isotropic and let $\mathbf{H}_1 := \mathbf{SO}(r)$. By Proposition 3.4, $R_{k_1/\mathbb{Q}}(\mathbf{H}_1)$ is \mathbb{Q} -isogenous to a \mathbb{Q} -subgroup of $R_{k_2/\mathbb{Q}}(\mathbf{SO}(q_2))$. Observe that

$$\dim \mathbf{SO}(q_2) = \frac{m(m-1)}{2} = \frac{(m-1)(m-2)}{2} + m < 2 \left(\frac{(m-1)(m-2)}{2} \right) = 2 \dim \mathbf{SO}(r).$$

Since M_1 and M_2 are \mathbb{R} -simple, we may apply Proposition 6.3 and the result follows. \square

Remark 6.5. By examining the proof, for $k(M_1)$ and $k(M_2)$ to be isomorphic, it is sufficient that M_1 and M_2 both contain, up to commensurability, the same totally geodesic subspace coming from a codimension one quadratic form.

7. TECHNICAL RESULTS: CONSTRUCTION OF SUBFORMS OF QUADRATIC FORMS

This section is dedicated to showing that over number fields, nonisogroupic forms cannot have the same isogroupy classes of subforms. Toward these ends, we construct proper quadratic subforms with very specific local properties that exploit the exceptional restrictions on the Hasse invariant in dimensions 1 and 2. The results of this section heavily rely upon the following fundamental lemma.

Lemma 7.1 (Square Existence Lemma). *Let*

- (i) k be a number field,
- (ii) S be a finite set of places of k , and
- (iii) for each $v \in S$, let α_v be a square class in k_v^\times .

Then there exists an $s \in k^\times$ for which $s \in \alpha_v$ for all $v \in S$.

Proof. Each nontrivial (resp. trivial) square class α_v corresponds to a unique quadratic (resp. trivial) extension L_v/k_v . By local class field theory, this corresponds to a character χ_v of k_v^\times of order 2 (resp. order 1). By the Grunwald–Wang Theorem [26, Chp VIII Thm 2.4], there exists

a character χ of $GL_1(\mathbb{A}_k)/GL_1(k)$ whose restriction to k_v^\times is χ_v , for all $v \in S$. Since $n = 2$ and $k[\zeta_2] = k$ is trivially cyclic, we may choose χ to have order 2. By global class field theory, this gives a quadratic extension L/k where $L = k(s)$. Then $s \in \alpha_v$ for all $v \in S$. \square

Constructing Nonrepresentable Subforms 1: Nonisogroupic at a real place. Let k be a number field and let q be an m -dimension quadratic form over k . If $v \in V_k$ is a real place, then we shall say q is **ordered at v** if the signature $(m_+^{(v)}, m_-^{(v)})$ of $q \otimes k_v$ satisfies $m \geq m_+^{(v)} \geq m_-^{(v)} \geq 0$. We call q **ordered** if it is ordered at all real places.

Lemma 7.2. *Let k be a number field and q a quadratic form over k . Then there exists an $a \in k^\times$ so that aq is ordered.*

Proof. Let $S \subset V_k$ denote the set of all real places and let $S_0 \subset S$ denote the set of all real places where q is not ordered. For each $v \in S$, let

$$\alpha_v = \begin{cases} -(k_v^\times)^2 & \text{if } v \in S_0, \\ (k_v^\times)^2 & \text{if } v \notin S_0. \end{cases}$$

By Lemma 7.1, there exists $a \in k^\times$ such that $a(k_v^\times)^2 = \alpha_v$ for all $v \in S$ and hence aq is ordered. \square

Recall that two quadratic forms over \mathbb{R} are \mathbb{R} -isogroupic if and only if they are similar.

Lemma 7.3. *Let q_1 and q_2 be nonisometric m -dimensional quadratic forms over \mathbb{R} with signatures (m_1, n_1) and (m_2, n_2) respectively such that $m_1 > m_2 \geq n_2 > n_1$. Then for all $j \in \mathbb{Z}_{\geq 1}$ such that*

$$n_1 + n_2 < j < m$$

there exists an isotropic j -dimensional form dividing q_2 that is not similar to a form dividing q_1 . Furthermore, this form can be realized by deleting $m - j$ entries in a diagonal representation of q_2 .

Proof. The idea of the proof is that we pick a subform r of q_2 such that neither r nor $-r$ divides q_1 . We may represent

$$q_1 = \underbrace{\langle a_1, \dots, a_{m_1} \rangle}_{>0} \underbrace{\langle a_{m_1+1}, \dots, a_m \rangle}_{<0} \quad \text{and} \quad q_2 = \underbrace{\langle b_1, \dots, b_{m_2} \rangle}_{>0} \underbrace{\langle b_{m_2+1}, \dots, b_m \rangle}_{<0},$$

with $a_i, b_j \in \mathbb{R}$. The desired subform may be obtained by deleting the first $m - j$ entries of q_2 , namely let

$$r := \langle b_{m-j+1}, b_{m-j+2}, \dots, b_{m-1}, b_m \rangle.$$

By construction, r has signature $(j - n_2, n_2)$ from which we can see that r is always isotropic and both

- $j - n_2 > n_1 + n_2 - n_2 = n_1$, and
- $n_2 > n_1$.

Hence neither r nor $-r$ is a subform of q_1 . \square

Remark 7.4. The more isotropic both forms are, the fewer subforms arise from this construction. In particular, there are no subforms precisely when m is even and the two forms have signatures

$$\left(\frac{m}{2} - 1, \frac{m}{2} + 1\right) \quad \text{and} \quad \left(\frac{m}{2}, \frac{m}{2}\right).$$

Lemma 7.5. *Let q_1 and q_2 be nonisometric m -dimensional quadratic forms over \mathbb{R} with signatures (m_1, n_1) and (m_2, n_2) respectively such that $m_1 > m_2 \geq n_2 > n_1 > 0$. Then for all $j \in \mathbb{Z}_{\geq 1}$ such that*

$$m_1 < j < m$$

there exists an isotropic j -dimensional form dividing q_1 that is not similar to a form dividing q_2 . Furthermore, this form can be realized by deleting $m - j$ entries in a diagonal representation of q_1 .

Proof. Again we may represent

$$q_1 = \underbrace{\langle a_1, \dots, a_{m_1} \rangle}_{>0} \underbrace{\langle a_{m_1+1}, \dots, a_m \rangle}_{<0} \quad \text{and} \quad q_2 = \underbrace{\langle b_1, \dots, b_{m_2} \rangle}_{>0} \underbrace{\langle b_{m_2+1}, \dots, b_m \rangle}_{<0},$$

with $a_i, b_j \in \mathbb{R}$. This time the desired subform may be obtained by deleting the last $m - j$ entries of q_1 , namely let

$$r := \langle a_1, a_2, \dots, a_j \rangle.$$

By construction, r has signature $(m_1, n_1 - m + j)$ from which we can see that r is always isotropic and by our initial assumptions, both $m_1 > m_2$ and $m_1 > n_2$. Hence neither r nor $-r$ is a subform of q_2 . \square

Remark 7.6. The more anisotropic q_1 is, the fewer subforms arise from this construction. In particular, there are no subforms arising from this construction precisely when $m_1 = m - 1$.

Combining Lemma 7.3 and Lemma 7.5 we obtain the following corollary.

Corollary 7.7. *Let q_1 and q_2 be nonisogroupic quadratic forms over \mathbb{R} of dimension $m \geq 5$. Then there exists an isotropic $(m - 1)$ -dimensional subform of one which is not similar to a subform of the other. Furthermore, this form can be realized by deleting one entry in a diagonal representation of either q_1 or q_2 .*

The bound $m \geq 5$ is tight since neither Lemma 7.3 nor Lemma 7.5 may be applied to the nonisometric 4-dimensional real forms q_1 and q_2 with signatures $(3, 1)$ and $(2, 2)$ respectively. It is not hard to see that every isotropic \mathbb{R} -subform of one is isogroupic to an \mathbb{R} -subform of the other.

Proposition 7.8. *Let k be a totally real number field and let $m \geq 5$. Suppose that q_1 and q_2 are ordered m -dimensional quadratic forms over k that are isotropic at precisely one real place, v_1 and v_2 , respectively, and q_{1,v_1} and q_{2,v_2} are not \mathbb{R} -isometric. Then, up to relabelling, there exists an $(m - 1)$ -dimensional quadratic k -subform r of q_1 that is not isogroupic to a subform of any twist of q_2 . Furthermore r can be chosen to be isotropic at the real places where q_1 is isotropic.*

Proof. We may replace q_2 with a twist such that q_1 and q_2 are isotropic at the same real place. The result then follows from Corollary 7.7. \square

Constructing Nonrepresentable Subforms 2: Isogroupic at all real places. In this section, we look at quadratic forms over k that are isogroupic at all infinite places but are not isogroupic at a finite place.

Theorem 7.9. *Let k be a number field and let $m = 2n + 1$ for $n \geq 2$. Suppose that q_1 and q_2 are nonisometric m -dimensional quadratic forms over k such that*

- (i) *at each infinite place v , $q_{1,v}$ and $q_{2,v}$ are ordered and isometric, and*
- (ii) *there is a finite place $v_0 \in V_k$ where:*
 - (a) $\det_{v_0} q_1 = 1 = \det_{v_0} q_2$,
 - (b) $c_{v_0}(q_1) \neq c_{v_0}(q_2)$.

Then there exists an $(m - 1)$ -dimensional quadratic subform r of q_1 that is not isogroupic to a subform of q_2 . Furthermore if q_1 is isotropic at a real place, then r can be chosen to be isotropic at that real place as well.

Proof. We construct r locally and then patch it together into a global form using the local-to-global results of Section 4. Let $S = \{v_0\} \cup \{\text{infinite real places of } k\}$ and for each $v \in S$, we pick square classes $\alpha_v \in k_v^\times / (k_v^\times)^2$ as follows:

- For v_0 , let α_{v_0} be such that $\alpha_{v_0} = (-1)^n$.
- For each infinite $v \in S$ let $\alpha_v = \det q_1(k_v^\times)^2 (= \det q_2(k_v^\times)^2)$.

By Lemma 7.1, we may choose an $s \in k^\times$ such that $s \in \alpha_v$ for all $v \in S$.

For each finite place $v \in V_k$, define t_v and r_v to be the quadratic k_v -forms with invariants:

$$\begin{aligned} \dim t_v &= 1 & \dim r_v &= m - 1 \\ \det t_v &= \frac{\det q_1}{s} & \det r_v &= s \\ c_v(t_v) &= 1 & c_v(r_v) &= c_v(q_1) \left(s, \frac{\det q_1}{s} \right)_v. \end{aligned}$$

Such forms exist by Theorem 4.5 (3).

For each infinite place $v \in V_k$, define the form t_v by:

$$t_v = \left\langle \frac{\det q_1}{s} \right\rangle.$$

At each complex place, t_v divides $q_1 \otimes k_v$. At each real place, q_1 and q_2 are ordered, and hence $t_v = \langle 1 \rangle$ is a subform of $q_1 \otimes k_v$. Therefore at each infinite place it makes sense to take the complement of t_v in $q_1 \otimes k_v$ and we may define the form r_v by

$$r_v = t_v^\perp.$$

At each complex place $v \in V_k$, we trivially have $c_v(r_v) = 1 = c_v(q_1)$. For each real $v \in V_k$, let $(m_+^{(v)}, m_-^{(v)})$ denote the signature of $q_1 \otimes k_v$. Observe that r_v has signature $(m_+^{(v)} - 1, m_-^{(v)})$, and hence is isotropic whenever $q_1 \otimes k_v$ is isotropic. Also note that at an isotropic real place,

$$c_v(r_v) = (-1)^{\frac{m_-^{(v)}(m_-^{(v)}-1)}{2}} = c_v(q_1).$$

We now show that for each place $v \in V_k$, $t_v \oplus r_v \cong q_1 \otimes k_v$. This is true by construction at the infinite places. When v is finite, we have

$$\begin{aligned} \dim(t_v \oplus r_v) &= 1 + (n - 1) = n = \dim(q_1 \otimes k_v), \\ \det(t_v \oplus r_v) &= (\det q_1 / s) s = \det(q_1 \otimes k_v), \end{aligned}$$

and by the product formula for the Hasse invariant

$$c(t_v \oplus r_v) = c(t_v)c(r_v) \left(\frac{\det q_1}{s}, s \right) = c_v(q_1) \left(\frac{\det q_1}{s}, s \right)^2 = c(q_1 \otimes k_v).$$

By Theorem 4.4 (3), they are isometric.

To build a global form, we must check that our forms satisfy the compatibility criteria of Theorem 4.7. Observe that $c_v(t_v) = 1$ for all $v \in V_k$ and hence $\prod_{v \in V_k} c_v(t_v) = 1$. By our choice of s , $\left(s, \frac{\det q_1}{s} \right)_v = 1$ at each infinite place, and hence

$$\begin{aligned} \prod_{v \in V_k} c_v(r_v) &= \left(\prod_{v \in V_k \text{ finite}} c_v(r_v) \right) \times \left(\prod_{v \in V_k \text{ real}} c_v(r_v) \right) \times \left(\prod_{v \in V_k \text{ complex}} c_v(r_v) \right) \\ &= \left(\prod_{v \in V_k \text{ finite}} c_v(q_1) \left(s, \frac{\det q_1}{s} \right)_v \right) \times \left(\prod_{v \in V_k \text{ real}} c_v(q_1) \right) \times \left(\prod_{v \in V_k \text{ complex}} c_v(q_1) \right) \\ &= \prod_{v \in V_k} c_v(q_1) \times \prod_{v \in V_k} \left(s, \frac{\det q_1}{s} \right)_v \\ (7.1) \quad &= 1. \end{aligned}$$

The final product is trivial because both the Hasse invariant and the Hilbert symbol of global objects satisfy the product formula.

By Theorem 4.7, there exist quadratic forms t and r over k such that for all $v \in V_k$, $t \otimes k_v \cong t_v$ and $r \otimes k_v \cong r_v$. Furthermore, for each $v \in V_k$, we have shown that $t_v \oplus r_v \cong q_1 \otimes k_v$ so by Theorem 4.6 we conclude $t \oplus r \cong q_1$, and hence r is a subform of q_1 .

Suppose that r' is isogroupic to r . Since $\mathbf{H} := \mathbf{SO}(r)$ is a group of type D_n over k , it determines the following invariants of r' :

- (i) $\dim(r') = 2n = \dim(r)$.
- (ii) $\text{disc}_v(r') = 1$ at precisely the places $v \in V_k$ where $\mathbf{H} \otimes k_v$ is a group of inner type (i.e., the $*$ -action is trivial). This means that $\text{disc}_v(r') = 1$ if and only if $\text{disc}_v(r) = 1$, or in other words, at the places where $\text{disc}_v(r) = 1$, then $\det r' = \det r$.
- (iii) $c_v(r') = c_v(r)$ at each place v where $\text{disc}_v(r) = 1$ (see equation (5.2)).

We now show that no subform of q_2 is isogroupic to r . Suppose that r' is isogroupic to r and there exists some form t' such that $r' \oplus t' \cong q_2$. It immediately follows that $\dim t' = 1$, $\det t' = \frac{\det q_2}{\det r'}$, and, by the exceptional restriction, $c(t') = 1$. Our choice of s implies:

$$\text{disc } r_{v_0} = (-1)^n \det r_{v_0} = (-1)^{2n} = 1,$$

hence $\det r_{v_0} = \det r'_{v_0}$ and $c(r_{v_0}) = c(r'_{v_0})$. Therefore:

$$\begin{aligned} c(q_{2,v_0}) &= c(r'_{v_0} \oplus t'_{v_0}) \\ &= c(r'_{v_0})c(t'_{v_0}) \left(\det r'_{v_0}, \frac{\det q_{2,v_0}}{\det r'_{v_0}} \right) \\ &= c(r_{v_0}) \left(\det r_{1,v_0}, \frac{\det q_{2,v_0}}{\det r_{v_0}} \right) \\ &= \left(c(q_{1,v_0}) \left(\det r_{v_0}, \frac{\det q_{1,v_0}}{\det r_{v_0}} \right) \right) \left(\det r_{v_0}, \frac{\det q_{2,v_0}}{\det r_{v_0}} \right) \\ &= c(q_{1,v_0}) \left(\det r_{1,v_0}, \frac{\det q_{1,v_0} \det q_{2,v_0}}{(\det r_{1,v_0})^2} \right) \\ &= c(q_{1,v_0}) (\det r_{1,v_0}, 1) \\ &= c(q_{1,v_0}). \end{aligned}$$

This contradicts our initial assumption that $c(q_{1,(j)}) \neq c(q_{2,v_0})$ and the result follows. \square

Example 7.10. Consider the following 5-dimensional quadratic forms over \mathbb{Q} :

$$q_1 = \langle 1, 1, 1, 1, -5 \rangle \quad \text{and} \quad q_2 = \langle 1, 1, 3, 3, -5 \rangle.$$

Observe that $\det q_1 = -5 = \det q_2$, which in \mathbb{Q}_3 is a square. Furthermore, a quick computation shows $c_3(q_1) = 1$ and $c_3(q_2) = -1$. By Theorem 7.9, there exists a 4-dimensional quadratic form $r \subset q_1$ so that $\mathbf{H} := \mathbf{SO}(r) \subset \mathbf{SO}(q_1)$ but \mathbf{H} is not \mathbb{Q} -isomorphic to a subgroup of $\mathbf{SO}(q_2)$. It is not hard to check that $r = \langle 1, 1, 1, -5 \rangle$ is such a form.

Lemma 7.11. *Let k_v be a nonarchimedean local field, q be a $2n$ -dimensional quadratic form over k_v , and r be a codimension one subform of q .*

- (i) *If $\mathbf{SO}(q)$ is of type ${}^1D_{n,n}^{(1)}$, then $\mathbf{SO}(r)$ is of type $B_{n-1,n-1}$.*
- (ii) *If $\mathbf{SO}(q)$ is of type ${}^1D_{n,n-2}^{(1)}$, then $\mathbf{SO}(r)$ is of type $B_{n-1,n-2}$.*

Proof. We show (i). Let $s = \det r$. A direct computation yields

$$\begin{aligned} c(r) &= c(q) \left(s, \frac{\det q}{s} \right) \\ &= (-1, -1)^{\frac{n(n-1)}{2}} (s, -\det q) \end{aligned}$$

$$\begin{aligned}
&= (-1, -1)^{\frac{(n-4)(n-1)}{2}} (s, -(-1)^n) \\
&= (-1, -1)^{\frac{(n-1)(n-4)}{2}} (-1, s)^{n-1}.
\end{aligned}$$

The result follows from equation (5.1). By inserting negative signs, (ii) follows analogously. \square

Theorem 7.12. *Let k be a number field and let $m = 2n \geq 4$. Suppose that q_1 and q_2 are nonisometric m -dimensional quadratic forms over k such that*

- (i) *at each infinite place v , $q_{1,v}$ and $q_{2,v}$ are ordered and isometric, and*
- (ii) *there is a finite place $v_0 \in V_k$ where*
 - (a) *$\text{disc}_{v_0}(q_1) = 1 = \text{disc}_{v_0}(q_2)$, and*
 - (b) *$c_{v_0}(q_1) = (-1, -1)^{\frac{n(n-1)}{2}} \neq -(-1, -1)^{\frac{n(n-1)}{2}} = c_{v_0}(q_2)$.*

Then there exists an $(m-1)$ -dimensional quadratic subform r of q_1 that is not isogroupic to a subform of q_2 . Furthermore if the q_1 is isotropic at a real place, then r can be chosen to be isotropic at that real place as well.

Proof. Again we are going to construct the desired forms locally and then use the existence, uniqueness, and local-to-global results of Section 4 to create the desired global forms. Let $S = \{v_0\} \cup \{\text{infinite real places of } k\}$ and for each $v \in S$, let $\alpha_v = \det q_1(k_v^\times)^2$. By Lemma 7.1, we choose an $s \in k^\times$ for which $s \in \alpha_v$ for all $v \in S$.

For each finite place $v \in V_k$, define t_v, r_v to be the quadratic k_v -forms with invariants given by:

$$\begin{aligned}
\dim t_v &= 1 & \dim r_v &= m-1 \\
\det t_v &= \frac{\det q_1}{s} & \det r_v &= s \\
c_v(t_v) &= 1 & c_v(r_v) &= c_v(q_1) \left(s, \frac{\det q_1}{s} \right)_v.
\end{aligned}$$

Such forms exist by Theorem 4.5 (3). For each infinite place $v \in V_k$, define t_v by:

$$t_v = \left\langle \frac{\det q_1}{s} \right\rangle.$$

At each complex place t_v divides $q_1 \otimes k_v$. By assumption, q_1 is ordered at each real place $v \in V_k$, and hence $t_v = \langle 1 \rangle$ is a subform of $q_1 \otimes k_v$. Therefore at each infinite place it makes sense to take the complement of t_v in $q_1 \otimes k_v$ and we may define forms r_v by

$$r_v = t_v^\perp.$$

At each complex place $v \in V_k$, we trivially have $c_v(r_v) = 1 = c_v(q_1)$. For each real $v \in V_k$, let $(m_+^{(v)}, m_-^{(v)})$ denote the signature of $q_1 \otimes k_v$. Observe that r_v has signature $(m_+^{(v)} - 1, m_-^{(v)})$, and hence is isotropic whenever $q_1 \otimes k_v$ is isotropic. At such an isotropic real place, $c_v(r_v) = c_v(q_1)$.

Just as in the proof of Theorem 7.9, we have:

- The families $\{t_v\}_{v \in V_k}$ and $\{r_v\}_{v \in V_k}$ satisfy the global compatibility conditions (see 7.1), and hence by Theorem 4.7, there exist quadratic forms t and r over k such that for all $v \in V_k$, $t \otimes k_v \cong t_v$ and $r \otimes k_v \cong r_v$.
- By Theorem 4.4 (3), $t_v \oplus r_v$ and $q_1 \otimes k_v$ are isometric at each place $v \in V_k$.
- By Theorem 4.6 we conclude $t \oplus r \cong q_1$, and hence r is a subform of q_1 .

Our assumptions on the Hasse invariant at v_0 and equation (5.2) imply that $\mathbf{SO}(q_{1,v_0})$ is of type ${}^1D_{n,n}^{(1)}$ while $\mathbf{SO}(q_{2,v_0})$ is of type ${}^1D_{n,n-2}^{(1)}$. By Lemma 7.11(i), $\mathbf{SO}(r)$ is the split group $B_{n-1,n-1}$ at v_0 , but by Lemma 7.11(ii), every codimension one subbform of q_2 yields the nonsplit group of type $B_{n-1,n-2}$. Hence no subform of q_2 is isogroupic to r . \square

Example 7.13. Consider the following 4-dimensional quadratic forms over \mathbb{Q} :

$$q_1 = \langle 1, 1, 5, -1 \rangle \quad \text{and} \quad q_2 = \langle 3, 3, 5, -1 \rangle.$$

Observe that $\det q_1 = -5 = \det q_2$, which in \mathbb{Q}_3 is a square. Hence these have discriminant 1 in \mathbb{Q}_3 . Furthermore, a quick computation shows $c_3(q_1) = 1$ and $c_3(q_2) = -1$. By Theorem 7.12, there exists a 3-dimensional quadratic form $r \subset q_1$ so that $\mathbf{H} := \mathbf{SO}(r) \subset \mathbf{SO}(q_1)$ but \mathbf{H} is not \mathbb{Q} -isomorphic to a subgroup of $\mathbf{SO}(q_2)$. It is not hard to check that $r = \langle 1, 1, -1 \rangle$ is such a form.

Theorem 7.14. *Let k be a number field and let $m = 2n \geq 6$. Suppose that q_1 and q_2 are nonisometric m -dimensional quadratic forms over k such that*

- (i) *at each infinite place v , $q_{1,v}$ and $q_{2,v}$ are ordered and isometric, and*
- (ii) *there is a finite place $v_0 \in V_k$ where:*
 - (a) $\text{disc}_{v_0} q_1 = 1$,
 - (b) $\text{disc}_{v_0} q_2 \neq 1$,
 - (c) $c_{v_0}(q_1) \neq c_{v_0}(q_2)(-1, \text{disc}(q_2))_{v_0}^{\frac{m-2}{2}}$

Then there exists an $(m-2)$ -dimensional quadratic subform r of q_2 that is not isogroupic to a subform of q_1 . Furthermore if q_2 is isotropic at a real place, then r can be chosen to be isotropic at that real place as well.

Proof. As we did in Theorems 7.9 and 7.12 we construct the desired forms locally and use the results of Section 4 to create global forms. Let $S = \{v_0\} \cup \{\text{infinite real places of } k\}$ and for each $v \in S$, we pick square classes $\alpha_v \in k_v^\times / (k_v^\times)^2$ as follows:

- For v_0 , let α_{v_0} be such that $\alpha_{v_0} = (-1)^{\frac{m-2}{2}} (k_{v_0}^\times)^2$.
- For infinite $v \in S$ let $\alpha_v = \det q_2 (k_v^\times)^2$.

By Lemma 7.1 we choose an $s \in k^\times$ for which $s \in \alpha_v$ for all $v \in S$.

For each finite place $v \in V_k$, define t_v, r_v to be the quadratic k_v -forms with invariants given by:

$$\begin{aligned} \dim t_v &= 2 & \dim r_v &= m-2 \\ \det t_v &= \frac{\det q_2}{s} & \det r_v &= s \\ c_v(t_v) &= 1 & c_v(r_v) &= c_v(q_2) \left(s, \frac{\det q_2}{s} \right)_v. \end{aligned}$$

We know such forms exist by Theorem 4.5 (3).

For each infinite place $v \in V_k$, define form t_v by:

$$t_v = \left\langle 1, \frac{\det q_2}{s} \right\rangle.$$

At each complex place t_v divides $q_2 \otimes k_v$. By assumption, q_2 is ordered at each real place $v \in V_k$, and hence $t_v = \langle 1, 1 \rangle$ is a subform of $q_2 \otimes k_v$. Therefore at each infinite place it makes sense to take the complement of t_v in $q_2 \otimes k_v$ and we may define forms r_v by

$$r_v = t_v^\perp.$$

At each complex place $v \in V_k$, we trivially have $c_v(r_v) = 1 = c_v(q_2)$. For each real $v \in V_k$, let $(m_+^{(v)}, m_-^{(v)})$ denote the signature of $q_2 \otimes k_v$. Observe that r_v has signature $(m_+^{(v)} - 2, m_-^{(v)})$, and hence is isotropic whenever $q_2 \otimes k_v$ is isotropic. Note that $c_v(r_v) = c_v(q_2)$.

We shall now show that for each place $v \in V_k$, $t_v \oplus r_v \cong q_2 \otimes k_v$. This is true by construction at the infinite places. Now suppose v is finite. Clearly

$$\dim(t_v \oplus r_v) = 1 + (n-1) = n = \dim(q_2 \otimes k_v),$$

$$\det(t_v \oplus r_v) = (\det q_2/s)s = \det(q_2 \otimes k_v),$$

and by the product formula for the Hasse invariant

$$c(t_v \oplus r_v) = c(t_v)c(r_v) \left(\frac{\det q_2}{s}, s \right) = c_v(q_2) \left(\frac{\det q_2}{s}, s \right)^2 = c(q_2 \otimes k_v),$$

and hence by Theorem 4.4 (3), they are isomorphic.

We now wish to build a global form, and hence must check that our forms satisfy the compatibility criteria of Theorem 4.7. Observe that $c_v(t_v) = 1$ for all $v \in V_k$ and hence $\prod_{v \in V_k} c_v(t_v) = 1$. Next observe that by our choice of s , $\left(s, \frac{\det q_2}{s}\right)_v = 1$ at each infinite place, and hence

$$\begin{aligned} \prod_{v \in V_k} c_v(r_v) &= \left(\prod_{v \in V_k \text{ finite}} c_v(r_v) \right) \times \left(\prod_{v \in V_k \text{ real}} c_v(r_v) \right) \times \left(\prod_{v \in V_k \text{ complex}} c_v(r_v) \right) \\ &= \left(\prod_{v \in V_k \text{ finite}} c_v(q_2) \left(s, \frac{\det q_2}{s} \right)_v \right) \times \left(\prod_{v \in V_k \text{ real}} c_v(q_2) \right) \times \left(\prod_{v \in V_k \text{ complex}} c_v(q_2) \right) \\ &= \prod_{v \in V_k} c_v(q_2) \times \prod_{v \in V_k} \left(s, \frac{\det q_2}{s} \right)_v \\ (7.2) \quad &= 1. \end{aligned}$$

Again, the final product is trivial because both the Hasse invariant and the Hilbert symbol of global objects satisfy product formulas.

By Theorem 4.7, there exist quadratic forms t and r over k such that for all $v \in V_k$, $t \otimes k_v \cong t_v$ and $r \otimes k_v \cong r_v$. Furthermore, for each $v \in V_k$, we have shown that $t_v \oplus r_v \cong q_2 \otimes k_v$ so by Theorem 4.6 we conclude $t \oplus r \cong q_2$, and hence r is a subform of q_2 .

Let $\mathbf{H} = \mathbf{SO}(r)$. We will show that $\mathbf{H} \not\subset \mathbf{G}_1 = \mathbf{SO}(q_1)$, and hence that there are no representatives r' of \mathbf{H} such that $r' \subset q_1$. Again \mathbf{H} is a group of type D_n over k . Let r' be any representative of \mathbf{H} . As in the proof of Theorem 7.9, the group \mathbf{H} determines the following invariants of r' :

- (i) $\dim(r') = 2n - 2 = \dim(r)$.
- (ii) $\text{disc}_v(r') = 1$ at precisely the places $v \in V_k$ where $\mathbf{H} \otimes k_v$ is a group of inner type (i.e., the $*$ -action is trivial). This means that $\text{disc}_v(r') = 1$ if and only if $\text{disc}_v(r) = 1$, or in other words, at the places where $\text{disc}_v(r) = 1$, then $\det r' = \det r$.
- (iii) $c_v(r') = c_v(r)$ at each place v where $\text{disc}_v(r) = 1$ (see equation (5.2)).

Let r' be any quadratic form satisfying these three properties. Suppose there exists some form t' such that $r' \oplus t' \cong q_1$. It follows that $\dim t' = 2$, $\det t' = \det q_1 / \det r'$.

Our choice of s implies that

$$\text{disc}_v(r) = (-1)^{(n-1)} \det r = (-1)^{2n-2} = 1.$$

Hence at v_0 , we have $\det r = \det r'$ and $c_{v_0}(r) = c_{v_0}(r')$. Furthermore we have

$$\begin{aligned} \det_{v_0} t' &= \frac{\det_{v_0} q_1}{\det_{v_0} r'} \\ &= \frac{(-1)^{\frac{m}{2}} \text{disc}(q_1)}{(-1)^{\frac{m-2}{2}} \text{disc}(r')} \\ &= \frac{(-1)^{\frac{m}{2}}}{(-1)^{\frac{m-2}{2}}} \\ &= -1, \end{aligned}$$

and thus by the exceptional restriction, $c_{v_0}(t') = 1$. The product formula at v_0 now yields the following contradiction:

$$\begin{aligned}
c_{v_0}(q_1) &= c_{v_0}(r' \oplus t') \\
&= c_{v_0}(r') \left(\det r', \frac{\det q_1}{\det r'} \right)_{v_0} \\
&= c_{v_0}(q_2) \left(\det r', \frac{\det q_2}{\det r'} \right)_{v_0} \left(\det r', \frac{\det q_1}{\det r'} \right)_{v_0} \\
&= c_{v_0}(q_2) \left((-1)^{\frac{m-2}{2}}, \frac{\det q_1 \det q_2}{(\det r')^2} \right)_{v_0} \\
&= c_{v_0}(q_2) \left((-1)^{\frac{m-2}{2}}, (-1)^{\frac{m}{2}} \det q_2 \right)_{v_0} \\
&= c_{v_0}(q_2) (-1, \text{disc}(q_2))_{v_0}^{\frac{m-2}{2}}.
\end{aligned}$$

Hence r is not isogroupic to a subform of q_1 , concluding the proof. \square

Example 7.15. Consider the following 6-dimensional quadratic forms over \mathbb{Q} :

$$q_1 = \langle 1, 1, 1, 3, 3, -1 \rangle \quad \text{and} \quad q_2 = \langle 1, 1, 1, 1, 1, -5 \rangle.$$

Observe that $\det q_1 = -1 \neq -5 = \det q_2$. Furthermore, $\text{disc}_3(q_1) = 1$, but $\text{disc}_3(q_2) = 5$ which is not a square in \mathbb{Q}_3 . Furthermore, a quick computation shows $c_3(q_1) = -1$ and $c_3(q_2) = 1$. By Theorem 7.14, there exists a 4-dimensional quadratic form $r \subset q_2$ so that $\mathbf{H} := \mathbf{SO}(r) \subset \mathbf{SO}(q_2)$ but \mathbf{H} is not \mathbb{Q} -isomorphic to a subgroup of $\mathbf{SO}(q_1)$. It is not hard to check that $r = \langle 1, 1, 1, -5 \rangle$ is such a form.

Constructing Subforms In Codimension > 2 . We have shown that given certain nonisometric forms, we may find codimension one or codimension two subforms of one that are not isogroupic to a subform of the other. We now show that this is the best we can hope for.

Proposition 7.16. *Let k be a number field and let q_1 and q_2 be m -dimensional quadratic forms over k , $m \geq 4$, which are isometric at each infinite place. If r is a j -dimensional subform of q_1 , where $0 < j < m - 2$, then r is also a subform of q_2 .*

Proof. For each finite $v \in V_k$, let t_v be the k_v form uniquely determined by

- $\dim t_v = n - m$,
- $\det t_v = \frac{\det q_2}{\det r}$, and
- $c_v(t_v) = c_v(q_2) c_v(r) \left(\det(r), \frac{\det(q_2)}{\det(r)} \right)_v$.

Such forms exist by Theorem 4.5 (3). Since $q_1 \otimes k_v$ and $q_2 \otimes k_v$ are isometric at each infinite $v \in V_k$, $r \otimes k_v$ is a subform of $q_2 \otimes k_v$, and it makes sense to take its complement. We define

- $t_v := (r \otimes k_v)^\perp$,

It follows that at each infinite place v ,

$$c_v(t_v) = c_v(q_2) c_v(r) \left(\det(r), \frac{\det(q_2)}{\det(r)} \right)_v.$$

To build a global form, we must check that our forms satisfy the compatibility criteria of Theorem 4.7. This can be seen with the following computation:

$$\begin{aligned}
\prod_{v \in V_k} c_v(t_v) &= \left(\prod_{v \in V_k} c_v(q_2) c_v(r) \left(\det(r), \frac{\det(q_2)}{\det(r)} \right)_v \right) \\
&= \left(\prod_{v \in V_k} c_v(q_2) \right) \times \left(\prod_{v \in V_k} c_v(r) \right) \times \left(\prod_{v \in V_k} \left(\det(r), \frac{\det(q_2)}{\det(r)} \right)_v \right) \\
&= 1.
\end{aligned}$$

The final product is trivial because both the Hasse invariant and the Hilbert symbol of global objects satisfy the product formula. By Theorem 4.7 there is a quadratic form t over k such that for all $v \in V_k$, $t \otimes k_v \cong t_v$. Furthermore, for each $v \in V_k$, $t_v \oplus r_v$ and $q_2 \otimes k_v$ have the same local invariants so by Theorem 4.4 they are isometric, and by Theorem 4.6 we conclude $t \oplus r \cong q_2$, and hence r is a subform of q_2 . \square

8. PROOFS OF THEOREMS B AND C

Proof of Theorem B. By Theorem A we may assume that $k(M_1)$ and $k(M_2)$ are isomorphic and let k be a fixed representative of this isomorphism class. We now prove the contrapositive. Suppose that M_1 and M_2 are not commensurable. If q_1 and q_2 are k -groups giving rise to the spaces M_1 and M_2 , respectively, then q_1 is not k -isometric to any twist of q_2 . By Lemma 7.2, we may assume that q_1 and q_2 are ordered and furthermore, we may replace q_2 with a twist so that q_1 and q_2 are isotropic at the same infinite place. Since $\mathbf{SO}(q_1)$ and $\mathbf{SO}(q_2)$ are not k -isomorphic, the Hasse principle for special orthogonal groups [29, pg. 348] implies that there exist finite places v_0 where $\mathbf{SO}(q_1, v_0)$ and $\mathbf{SO}(q_2, v_0)$ are not k_{v_0} -isomorphic, and hence the forms q_1 and q_2 are not isometric over k_{v_0} .

First suppose m is odd. By Lemma 7.1, we may replace q_1 and q_2 with similar forms as necessary to guarantee that $\det_{v_0}^{(j)} q_1 = \det_{v_0} q_2 = 1$ while not altering the signatures at the infinite places. Hence $c_{v_0}(q_1) \neq c_{v_0}(q_2)$, and by Theorem 7.9 the result follows.

Now suppose $m = 2n$ is even. If $\det q_1 = \det q_2$ but $\text{disc}_{v_0}(q_i) \neq 1$, then by Lemma 4.3 and Lemma 7.1, we may replace q_2 with a similar form while not altering the signatures at the infinite place and for which $c_{v_0}(q_1) = c_{v_0}(q_2)$. This would imply q_1 and q_2 are isomorphic over k_{v_0} , contradicting our choice of v_0 . Hence if $\det q_1 = \det q_2$, then after possibly relabeling, their invariants must satisfy both of the following:

- (i) $\text{disc}_{v_0}(q_1) = 1 = \text{disc}_{v_0}(q_2)$, and
- (ii) $c_{v_0}(q_1) = (-1, -1)_{v_0}^{\frac{n(n-1)}{2}} \neq (-1, -1)_{v_0}^{\frac{n(n-1)}{2}} = c_{v_0}(q_2)$.

By Theorem 7.12 the result follows. Otherwise, if $\det q_1 \neq \det q_2$, then, after possible relabeling, we have $\text{disc}_{v_0} q_1 = 1$ and $\text{disc}_{v_0} q_2 \neq 1$. Furthermore, if $c_{v_0}(q_1) = c_{v_0}(q_2)(-1, \text{disc}(q_2))_{v_0}^{\frac{m-2}{2}}$, then we will replace q_2 with a similar form in the following way. Let $S = \{v_0\} \cup \{\text{infinite real places of } k\}$ and for each $v \in S$, we pick a square class $\alpha_v \in k_v^\times / (k_v^\times)^2$ as follows:

- at v_0 , $(\alpha_{v_0}, \text{disc}(q_2))_{v_0} = -1$ (note that such a class exists by the nondegeneracy of the Hilbert symbol and the fact that $\text{disc}(q_2) \neq 1$), and
- for all $v \in S$ real, α_v is trivial.

By Lemma 4.3, it follows that $c_{v_0}(\lambda q_2) = -c_{v_0}(q_2)$ and replacing q_2 by λq_2 , it follows that $c_{v_0}(q_1) \neq c_{v_0}(q_2)(-1, \text{disc}(q_2))_{v_0}^{\frac{m-2}{2}}$. By Theorem 7.14 the result follows. \square

Since commensurable spaces are length-commensurable, we have the following corollary.

Corollary 8.1. *Let M_1 and M_2 be \mathbb{R} -simple arithmetic locally symmetric spaces coming from quadratic forms of dimension ≥ 5 . If $\mathbb{Q}TG(M_1) = \mathbb{Q}TG(M_2)$, then $\mathbb{Q}L(M_1) = \mathbb{Q}L(M_2)$.*

Interestingly, Corollary 8.1 says that the set of totally geodesic subspaces determines the rational multiples of the lengths of all closed geodesics, even though there can exist closed geodesics that do not lie in any proper nonflat totally geodesic subspace. Upon specializing to \mathbb{R} -rank one spaces, Theorem C follows from Theorem B. Furthermore, unravelling the proof of Theorem B we see that we can tell apart noncommensurable spaces of type B_n by solely looking at codimension one subforms, yielding the following theorem.

Theorem 8.2. *Let M_1 and M_2 be even dimensional arithmetic hyperbolic n -orbifolds, $n \geq 4$. Suppose every codimension one, totally geodesic subspace in M_1 is commensurable to a codimension one totally geodesic subspace in M_2 and vice versa. Then M_1 and M_2 are commensurable.*

9. HYPERBOLIC SUBSPACE DICHOTOMY AND OTHER APPLICATIONS

In Construction 4.11 we showed that quadratic subforms give totally geodesic subspaces called subform subspaces. We now show that in the case of standard arithmetic hyperbolic orbifolds, these are the only finite volume totally geodesic subspaces.

Proposition 9.1. *If M is a standard arithmetic hyperbolic n -orbifold, $n \geq 4$, and $N \in \mathbb{Q}TG(M)$ then (i) $k(N) = k(M)$ and (ii) N is a subform subspace.*

Proof. By assumption, $M = M_q$ where (V, q) is a quadratic m -space, $m = n + 1 \geq 5$, over a totally real number field k with a unique real place v where q is isotropic. Let $\mathbf{G} = \mathbf{SO}(V, q)$ and denote k_v by \mathbb{R} . Let $H \subset G := \mathbf{G}(\mathbb{R})$ be the connected semisimple Lie subgroup giving rise to N . Since M_q is hyperbolic, it follows that $H = \mathbf{H}(\mathbb{R})^\circ$ where $\mathbf{H} = \mathbf{SO}(W', r')$ for some \mathbb{R} -subspace $W' \subset V_{\mathbb{R}}$ and r' the restriction of $q_{\mathbb{R}}$ to W' . Let $L \subset V$ be an \mathcal{O}_k -lattice and let G_L be its stabilizer in G . Since $\Lambda := G_L \cap H$ is a lattice in H , it is Zariski-dense in \mathbf{H} . It follows that the \mathbb{R} -span of $L \cap W'$ must be all of W' . Let W denote the k -span of $L \cap W'$ and let r be the restriction of q to W . Then $N = N_r$ and the result follows. \square

Proof of Theorem D. If M_1 and M_2 share a single finite volume totally geodesic subspace, Proposition 9.1 implies $k(M_1)$ and $k(M_2)$ are isomorphic. Let k be a fixed representative of the isomorphism class of $k(M_1)$ and $k(M_2)$. We may now choose quadratic forms q_1 and q_2 over k that are isotropic at the same real place and that give rise to M_1 and M_2 , respectively. The result then follows by Proposition 7.16. \square

In particular, since all noncompact arithmetic hyperbolic n -orbifolds, $n \geq 4$, come from $k = \mathbb{Q}$, we have the following corollary.

Corollary 9.2. *Let M_1 and M_2 be n -dimensional ($n \geq 4$) noncompact, standard arithmetic hyperbolic orbifolds. Then, up to commensurability, M_1 and M_2 have the exact same collection of finite volume totally geodesic subspaces of noncompact type of codimension > 2 .*

Recent work by McReynolds [24] shows that certain noncommensurable arithmetic manifolds arising from the semisimple Lie groups of the form $(\mathbf{SL}_d(\mathbb{R}))^r \times (\mathbf{SL}_d(\mathbb{C}))^s$ have the same commensurability classes of totally geodesic surfaces coming from a fixed field. An immediate consequence of our work above proves the following

Corollary 9.3. *For each $n \geq 4$, there exist noncommensurable, standard arithmetic, hyperbolic n -orbifolds M_1 and M_2 that have the same commensurability classes of totally geodesic surfaces.*

We conclude this section by addressing the following question was posed to us by Jean-François Lafont: *If M_1 and M_2 are good Riemannian orbifolds, when is it the case that $\mathbb{Q}TG(M_1) \subset \mathbb{Q}TG(M_2)$ implies $M_1 \subset M_2$?* We can answer this question for standard arithmetic hyperbolic orbifolds.

Proposition 9.4. *Let M_1 and M_2 be standard arithmetic hyperbolic spaces.*

- (i) *If $3 \leq \dim M_1 \leq \dim M_2 - 3$ and $\mathbb{Q}TG(M_1) \subset \mathbb{Q}TG(M_2)$, then, M_1 is commensurable to a totally geodesic subspace of M_2 .*
- (ii) *If $3 \leq \dim M_2 - 2 \leq \dim M_1$, there exist examples for which $\mathbb{Q}TG(M_1) \subset \mathbb{Q}TG(M_2)$ but M_1 is not commensurable to a totally geodesic subspace of M_2*

Proof. We begin by showing part (i). By assumption, every totally geodesic surface of M_1 is totally geodesic in M_2 . Proposition 9.1 implies that $k(M_1) = k(M_2) =: k$. Let q_i be quadratic forms over k which give rise to M_i that are isotropic at the same real place of k . Then by Proposition 7.16, it follows that q_1 is a subform of q_2 and (i) follows. Part (ii) follows from Example 9.5 below. \square

Example 9.5. Consider following quadratic forms over \mathbb{Q} described in Example 7.15:

$$q_1 = \langle 1, 1, 1, -5 \rangle \quad \text{and} \quad q_2 = \langle 1, 1, 1, 3, 3, -1 \rangle.$$

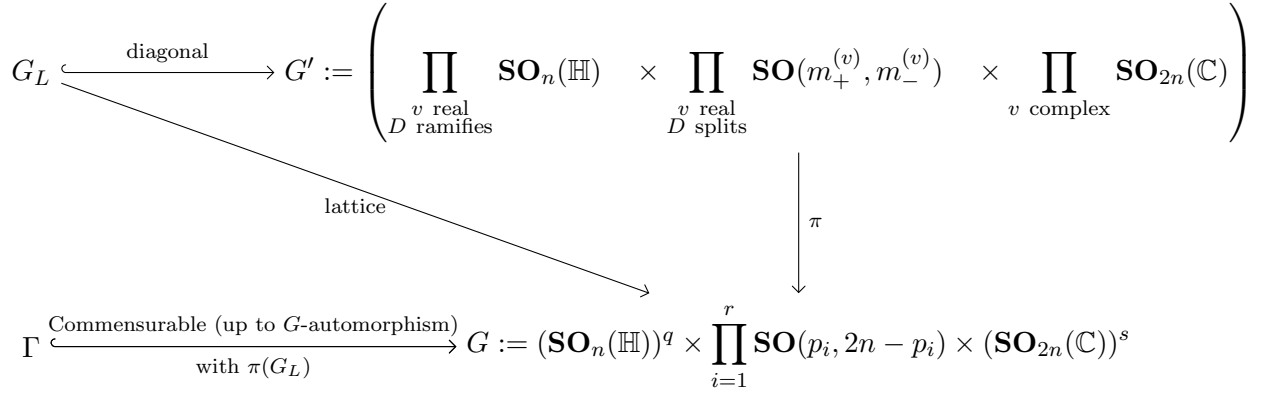
By Theorem 7.14 the 3-dimensional hyperbolic space M_{q_1} is not commensurable to a totally geodesic subspace of the 5-dimensional space M_{q_2} , yet by Proposition 7.16, they contain precisely the same totally geodesic surfaces.

10. NONSTANDARD ARITHMETIC HYPERBOLIC MANIFOLDS AND PROOFS OF THEOREMS E AND F

The arithmetic lattices in groups of type D_n that arise from skew Hermitian forms over division algebras over number fields we call **nonstandard lattices**. For the algebraic theory of skew Hermitian forms, we refer the reader to [33]. In this section $m = 2n \geq 4$ and \mathbb{H} denotes Hamilton's quaternions over \mathbb{R} .

Construction 10.1. Fix the following notation:

- (i) k is a number field with infinite places V_k^∞ ;
- (ii) D is a quaternion division algebra with center k ;
- (iii) (V, h) is an n -dimensional skew Hermitian space over D , $n \geq 2$;
- (iv) $\mathbf{G} := \mathbf{SU}(V, h)$ is the absolutely almost simple k -group defined by (V, h)
- (v) For each $v \in V_k^\infty$, $V_{k_v} := V \otimes_k k_v$, $q_v := q \otimes k_v$, and \mathbf{G}_v is the algebraic k_v -group $\mathbf{SU}(V_{k_v}, q_v)$
 - If v is real, and D ramifies over k_v , then $\mathbf{G}_v(k_v) \cong \mathbf{SO}_n(\mathbb{H})$.
 - If v is real, and D splits over k_v , then $\mathbf{G}_v(k_v) \cong \mathbf{SO}(m_+^{(v)}, m_-^{(v)})$.
 - If v is complex, then $\mathbf{G}_v(k_v) \cong \mathbf{SO}_{2n}(\mathbb{C})$.
 - q is the number of real places where D ramifies,
 - r is the number of real places where D splits and h is isotropic,
 - s is the number of complex places, and
 - $p_i = m_+^{(v_i)}$ where $\{v_1, \dots, v_r\}$ is the set of real places where D splits and h is isotropic;
- (vi) $G' := (R_k/\mathbb{Q}\mathbf{G})(\mathbb{R})$, G is the projection of G' and onto its noncompact factors, and $\pi : G' \rightarrow G$ is the projection map. Observe that G is a semisimple Lie group with no compact factors and is \mathbb{R} -simple when $q + r + s = 1$.
- (vii) \mathcal{O}_D is an order in D , $L \subset V$ is an \mathcal{O}_D -lattice, and $G_L := \{T \in \mathbf{G}(k) \mid T(L) \subset L\}$, which is a discrete subgroup of G'
- (viii) $\Gamma \subset G$ is commensurable up to G -automorphism with $\pi(G_L)$. Then Γ is said to be a **nonstandard arithmetic lattice of G** . Figure 10.1 summarizes this construction.
- (ix) $K \subset G$ is a maximal compact subgroup and $M_\Gamma := \Gamma \backslash G/K$. Then
 - (a) M_Γ is a **nonstandard arithmetic locally symmetric space of D_n** ,

FIGURE 10.1. Construction of Nonstandard arithmetic lattices in G .

- (b) $k(M_\Gamma) := k$ is the **field of definition** of M_Γ , and
- (c) $D(M_\Gamma) := D$ is the **algebra of definition** of M_Γ .

A choice of another order in D and another lattice in V will produce a space commensurable with M_L . Hence choosing h determines a commensurability class that we denote by M_h . More on this construction can be found in [18, §2]. Just as for quadratic forms (Construction 4.11), skew Hermitian subforms $h' \subset h$ give rise to commensurability classes totally geodesic subspaces $N_{h'} \subset M_h$, which we also call subform subspaces.

Definition 10.2. We call (k, D, h) an **admissible hyperbolic triple** if M_h is a commensurability class of hyperbolic orbifolds, that is to say,

- (i) k is totally real (i.e., no $\mathbf{SO}_{2n}(\mathbb{C})$ terms),
- (ii) D splits at all real places (i.e., no $\mathbf{SO}_m(\mathbb{H})$ terms),
- (iii) h is anisotropic at all but one real place, v_0 (i.e., only one $\mathbf{SO}(p_i, 2n - p_i)$ term),
- (iv) $q \otimes k_{v_0}$ has signature $(2n - 1, 1)$ or $(1, 2n - 1)$. (i.e., $G \cong \mathbf{SO}(2n - 1, 1)$).

Proof of Theorem E. If $\dim M_1 \neq \dim M_2$, it is not hard to find a subform subspace of one that is not commensurable to a proper totally geodesic subspace of the other, so suppose $\dim M_1 = \dim M_2$. Then there exists a $2n$ -dimensional quadratic form over a number field k_1 and an n -dimensional skew Hermitian form over a number field k_2 giving rise to M_1 and M_2 respectively. Furthermore, as we have already seen, we may choose a codimension one subform $r \subset q$ that is isotropic at the same real places as q . Suppose that $N_r \in \mathbb{Q}TG(M_2)$. By Proposition 3.4, $R_{k_1/\mathbb{Q}}\mathbf{SO}(r)$ is \mathbb{Q} -isogenous to a \mathbb{Q} -subgroup of $R_{k_2/\mathbb{Q}}\mathbf{SU}(h)$. Observe that, when $n \geq 4$,

$$\dim \mathbf{SU}(h) = n(2n - 1) < (2n - 1)(2n - 2) = 2 \left(\frac{(2n - 1)(2n - 2)}{2} \right) = 2 \dim \mathbf{SO}(r),$$

hence by Proposition 6.3, k_1 and k_2 are isomorphic. Let k be a fixed representative of this isomorphism class and replace h with a twist so that q and h are forms over k that are isotropic at the same infinite place. If $v \in V_k$ is a finite place where D ramifies, then

$$\text{rank}_{k_v}(\mathbf{SU}(h)) \leq \frac{n}{2} \leq n - 2 \leq \text{rank}_{k_v}(\mathbf{SO}(r)).$$

By local rank considerations $\mathbf{SO}(r)$ cannot be a subgroup of $\mathbf{SU}(h)$. □

Proof of Theorem F. Let $\mathbf{G}_i = \mathbf{SU}(h_i)$, $i = 1, 2$, be groups giving rise to M_i , where h_i is an n -dimensional skew Hermitian form over D_i . Let r be an $(n - 1)$ -dimensional Hermitian subform of h_1 which is isotropic at the real place h_1 is isotropic. Let $\mathbf{H}_1 := \mathbf{SU}(r)$. By assumption, the subform subspace $N_r \in \mathbb{Q}TG(M_2)$. By Proposition 3.4, $R_{k_1/\mathbb{Q}}\mathbf{H}_1$ is \mathbb{Q} -isogenous to a \mathbb{Q} -subgroup

of $R_{k_2/\mathbb{Q}}\mathbf{G}_2$. When $n \geq 4$,

$$\dim \mathbf{SU}(h_2) = n(2n - 1) < 2(n - 1)(2n - 3) = 2 \dim \mathbf{SU}(r),$$

and hence by Proposition 6.3, k_1 and k_2 are isomorphic. Let k be a fixed representative of this isomorphism class and replace h_2 with a twist so that h_1 and h_2 are forms over k that are isotropic at the same infinite place and D_1 and D_2 are quaternion algebras over k . Suppose that D_1 and D_2 are not isomorphic. Then there is a finite place $v \in V_k$ where one splits and the other ramifies. After relabeling if necessary, we may assume D_1 splits and D_2 ramifies. When $n \geq 4$,

$$\mathrm{rank}_{k_v}(\mathbf{G}_2) \leq \frac{n}{2} \leq n - 2 \leq \mathrm{rank}_{k_v}(\mathbf{H}).$$

Hence again by local rank considerations \mathbf{H} cannot be a subgroup of \mathbf{G}_2 and the result follows. \square

Question 4. *Let M_1 and M_2 be \mathbb{R} -simple, nonstandard arithmetic, locally symmetric spaces. Does $\mathrm{QTG}(M_1) = \mathrm{QTG}(M_2)$ imply M_1 and M_2 are commensurable?*

Question 4 remains open. The primary obstacle to answering this question is the lack of local and global existence theorems for skew Hermitian forms over division algebras. Answering this would complete the analysis of the rational totally geodesic spectrum for \mathbb{R} -simple arithmetic spaces of type D_n for all $n \geq 2$ not arising from triality in D_4 .

ACKNOWLEDGMENTS

We thank Matthew Stover for suggesting this problem and for countless helpful conversations. The author was supported by the NSF RTG grant 1045119. We would like to also thank Jean-François Lafont, Lucy Lifschitz, Benjamin Linowitz, D. B. McReynolds, and Ralf Spatzier for their interest in this project and for many valuable and interesting discussions.

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APPENDIX A. MACLACHLAN'S THEOREM: PARAMETRIZING COMMENSURABILITY CLASSES

In this section, we show how the techniques of Section 5 may be used to parametrize commensurability classes of even dimensional arithmetic hyperbolic orbifolds. In so doing, we provide an alternate proof of the results of Maclachlan [21], whose proof uses techniques from the theory of quaternion algebras and Clifford algebras.

Theorem A.1 (Maclachlan [21] Theorem 1.1). *The commensurability classes of arithmetic subgroups of $\text{Isom}(\mathbb{H}^{2n})$, $n \geq 1$, are parametrized for each totally real number field $k \subset \mathbb{R}$ by sets*

$\{\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_r\}$ of prime ideals in the ring of integers \mathcal{O}_k where

$$(A.1) \quad r \equiv \begin{cases} 0 \pmod{2} & \text{if } n \equiv 0 \pmod{4}, \\ [k : \mathbb{Q}] - 1 \pmod{2} & \text{if } n \equiv 1 \pmod{4}, \\ [k : \mathbb{Q}] \pmod{2} & \text{if } n \equiv 2 \pmod{4}, \\ 1 \pmod{2} & \text{if } n \equiv 3 \pmod{4}. \end{cases}$$

A place $v \in V_k$ is called **dyadic** if k_v is nonarchimedean with residue field of characteristic 2.

Lemma A.2. *Let k be a totally real number field and define*

$$\delta(k) := \left\{ \text{number of dyadic places where } \left(\frac{-1, -1}{\mathbb{Q}} \right) \text{ ramifies} \right\}.$$

Then $\delta(k) \equiv [k : \mathbb{Q}] \pmod{2}$.

Proof. Over \mathbb{Q} , Hamilton's quaternions ramify at precisely 2 and ∞ . Hence over k , Hamilton's quaternions ramify at precisely $\delta(k)$ places over 2, $[k : \mathbb{Q}]$ places over ∞ , and nowhere else. Since a quaternion algebra ramifies at an even number of places, the result follows. \square

Proof of Theorem A.1. We shall show that k -isomorphism classes of groups giving rise to standard arithmetic hyperbolic manifolds are parametrized by sets of the form $(v, \{\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_r\})$ where $v \in V_k$ is a real place and $\{\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_r\}$ is a set of prime ideals satisfying (A.1). The theorem then follows from [30, Prop 2.5] and our remarks after Construction 4.8 regarding twists of quadratic forms.

By Proposition 5.4, similarity classes of $(2n+1)$ -dimensional quadratic forms over k parametrize groups of type B_n over k . Picking the determinant 1 representative of each similarity class, the set

$$\mathcal{F} := \{q \mid \dim q = 2n+1, \det q = 1, \text{ and } (k, q) \text{ is an admissible hyperbolic pair}\}$$

parametrizes k -isomorphism classes of groups giving rise to arithmetic hyperbolic $2n$ -orbifolds. Let v_1, \dots, v_ℓ denote the real embeddings of k . For a fixed v_i , for $1 \leq i \leq \ell$, let

$$\mathcal{F}_i := \{q \in \mathcal{F} \mid q \text{ is isotropic at } v_i\}.$$

For $q \in \mathcal{F}_i$, the fact that $\det q = 1$ now implies that q has signature $(1, 2n)$ at v_i and signature $(2n+1, 0)$ at all other real places. A basic computation shows that the Hasse invariants at the real places are

$$c_{v_j}(q) = \begin{cases} (-1)^n & i = j \\ 1 & i \neq j. \end{cases}$$

Let $V_k^s = \{v \in V_k \mid (-1, -1)_v = +1\}$ and $V_k^r = \{v \in V_k \mid (-1, -1)_v = -1\}$. These sets correspond to the finite places where Hamilton's quaternions split and ramify, respectively. For $q \in \mathcal{F}_i$, let $e_s(q)$ (resp. $e_r(q)$) denote the number of finite places in V_k^s (resp. V_k^r) where $\mathbf{SO}(q)$ is not split. Clearly $r(q) := e_s(q) + e_r(q)$ is the total number of finite places where $\mathbf{SO}(q)$ is not split. (Note that this is always finite because any k -group is quasi-split at all but finitely many places and quasi-split groups of type B_n are split.)

Since q has determinant 1, (5.1) may be simplified to state that $\mathbf{SO}(q)$ splits over v if and only if $c_v(q) = (-1, -1)_v^{\frac{n(n-3)}{2}}$. Let $f_s(q)$ (resp. $f_r(q)$) denote the number of finite places v in V_k^s (resp. V_k^r) where $c_v(q) = -1$. If as in Lemma A.2, $\delta(k)$ is the number of dyadic places where $\left(\frac{-1, -1}{\mathbb{Q}} \right)$ ramifies, then it follows that:

- $f_s(q) = e_s(q)$, and
- $f_r(q) = \begin{cases} e_r(q) & \text{if } n \equiv 0, 3 \pmod{4}, \\ \delta(k) - e_r(q) & \text{if } n \equiv 1, 2 \pmod{4}. \end{cases}$

By Theorem 4.7, the local Hasse invariants of q must satisfy the compatibility condition $\prod_{v \in V_k} c_v(q) = 1$. It follows that

$$(-1)^n (-1)^{f_s(q)} (-1)^{f_r(q)} = 1$$

and hence

$$(A.2) \quad n + f_s(q) + f_r(q) \equiv 0 \pmod{2}.$$

We now have the following four cases:

- **Case 1:** $n \equiv 0 \pmod{4}$
Equation (A.2) immediately gives $r(q) \equiv 0 \pmod{2}$.
- **Case 2:** $n \equiv 1 \pmod{4}$
Equation (A.2) gives

$$n + e_s(q) + \delta(k) - e_r(q) \equiv 0 \pmod{2}.$$

By Lemma A.2 and simplifying,

$$1 + e_s(q) + [k : \mathbb{Q}] - e_r(q) \equiv 0 \pmod{2},$$

and hence

$$r(q) \equiv [k : \mathbb{Q}] - 1 \pmod{2}.$$

- **Case 3:** $n \equiv 2 \pmod{4}$
Again using Lemma A.2, equation (A.2) gives

$$0 + e_s(q) + [k : \mathbb{Q}] - e_r(q) \equiv 0 \pmod{2},$$

and hence

$$r(q) \equiv [k : \mathbb{Q}] \pmod{2}.$$

- **Case 4:** $n \equiv 3 \pmod{4}$
Equation (A.2) immediately gives $r(q) \equiv 1 \pmod{2}$.

We conclude that every form $q \in \mathcal{F}$ determines a set $(v_q, \{v_1, v_2, \dots, v_{r(q)}\})$ where v_q is the unique real place where q is isotropic, $\{v_1, v_2, \dots, v_{r(q)}\}$ is the set of finite places where $\mathbf{SO}(q)$ is not split over k_v , and $r(q)$ satisfies equation (A.1). Furthermore, by the local-to-global uniqueness of Theorem 4.4, no two forms $q, q' \in \mathcal{F}$ determine the same set of places.

Lastly we show that any collection $(v_0, \{v_1, v_2, \dots, v_r\})$ where $v_0 \in V_k$ is a real place, $\{v_1, v_2, \dots, v_r\}$ is a set of finite places, and r satisfies equation (A.1), determines a form in \mathcal{F} . Let $\{q_v\}_{v \in V_k}$ be a family of $(2n+1)$ -dimensional forms of determinant 1 satisfying the following:

- q_{v_0} has signature $(1, 2n)$,
- q_v has signature $(2n+1, 0)$ at all other real places,
- for $v \in V_k$ finite, $\mathbf{SO}(q_v)$ is not split if and only if $v \in \{v_1, v_2, \dots, v_r\}$, and hence $c_v(q_v)$ is determined by equation (5.1).

The above computations show that this family satisfies the compatibility condition of Theorem 4.7, and hence there exists a global form $q \in \mathcal{F}$ with localizations q_v . By Construction 4.8, we obtain a commensurability class and the result follows. \square

With proper modification, these techniques may be used to rederive Maclachlan's parametrization of commensurability classes of odd dimensional standard arithmetic hyperbolic orbifolds [21, Cor. 7.5]. Furthermore, with additional modifications, these techniques are generalizable to give parametrizations of commensurability classes of standard arithmetic lattices in groups of type B_n and D_n .